

HALF-SHIFTED YOUNG DIAGRAMS AND HOMOLOGY OF REAL GRASSMANNIANS

JORDAN LAMBERT SILVA AND LONARDO RABELO

ABSTRACT. Let G be a classical split real Lie group of type B , C , or D and P a maximal parabolic subgroup of G . The homogeneous space G/P is a minimal flag manifold of G and may be realized, respectively, as an even orthogonal, isotropic and odd orthogonal real Grassmannian. In this paper we compute the cellular \mathbb{Z} -homology group of such manifolds by a Lie theoretic approach which naturally furnish them with a cellular structure given by the Schubert varieties. The results are well understood in terms of what we call half-shifted Young diagrams whose construction is based on the the works of Buch-Kresch-Tamvakis, Ikeda-Naruse and Graham-Kreiman.

INTRODUCTION

Let G be a classical split real Lie group of type B , C , or D and let \mathfrak{g} be its split real semi-simple Lie algebra. Let P_Θ be a maximal parabolic subgroup of G of type Θ given as the complementary set of a root α_k of the Dynkin diagram of \mathfrak{g} . The homogeneous space G/P_Θ is called a minimal flag manifold of G . In the context of type B (resp. D), G/P_Θ may be realized as the orthogonal Grassmannians, i.e., submanifolds of isotropic subspaces in an odd (resp. even) dimensional vector space equipped with a non-degenerate symmetric bilinear form. In the context of type C , G/P_Θ may be realized as the isotropic Grassmannians, i.e., submanifolds of isotropic subspaces in an even dimensional vector space equipped with a symplectic form. In this paper we compute the cellular \mathbb{Z} -homology group of such manifolds developing the half-shifted diagrams that appears as a fundamental tool to describe the cellular structure given by the Schubert varieties.

In the general context of flag manifolds, the topology of the complex flag manifolds of semi-simple Lie groups is well developed and also a very ramified area of research nowadays (see Brion [2]). Yet the first results about the topology of real flag manifolds dates to 1970's by Burghelea–Hangan–Moscovici–Verona [5]. In the 1980's, the work of Duistermaat–Kolk–Varadarajan [8] computes the mod 2 homology. Only in the 1990's a complete description of its integral homology was done by Kocherlakota [14]. The Morse Theory appears as the main tool of these works. The first explicit cellular approach was described by Wiggerman [23] to get the fundamental groups of the real flag manifolds. Those ideas were generalized in Rabelo-San Martín [18] where a n -dimensional cellular version is obtained to improve the integral homology computations elucidating some ambiguities left in

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[14]. A more sophisticated point of view is the infinite dimensional representation theory of the real semi-simple Lie group described by Casian-Stanton [7]. The Euler Characteristic was addressed in the paper by Altomani-Medori-Nacinovicht [1]. A complete description of the orientability was described by Patrão-San Martin-Santos-Seco [15].

Notice that in the real context there are cells in all dimensions while in the complex one it appears only in even dimensions and hence torsion may occur in the real case whilst in the complex case the homology boundary maps are trivial elsewhere. Furthermore, the description of the cohomology ring over the integers of real flag manifolds is open in general. Some results have been done for the classical real Grassmannians, i.e., Grassmannians of type A by Casian-Kodama [6]. In this context, the homology groups were described by Burghelea-Hangan-Moscovici-Verona in the book [5]. Now, if we consider other Grassmannians (isotropic and orthogonal), the only specific results are the computations of its Euler Characteristic ([1]) and its orientability ([15]). In Rabelo [19], we have a first step towards solving the problem, whose result was obtained only for the maximal case.

It is remarkable that the Young diagrams has been largely used in the context of cohomology rings of complex Grassmannians mainly because its proven product properties related to the comohology rings of such manifolds. Instead, in the former paper of the second author [19], the strict Young diagrams were the main tool to deal with homology of some real Grassmannians. In this work, we extend those ideas about using adapted Young Diagrams to lead with the additive structure of the homology. Since the homology groups of real flag manifolds are completely described by what is called by Ikeda-Naruse as the beta-sequence of a Weyl group element, one of the main contribution of this paper lies in the presentation of the beta-sequences of the Weyl group elements that parametrizes all Grassmannians of types B,C and D in terms of the half-shifted Young diagrams. For type A, this was done by Ikeda-Naruse [12] and also by Graham-Kreiman [9]. We also hope that these diagrams may provide a way to get some results about the cohomology in the same sense that was done by Casian-Kodama [6] for type A. We now give a very brief introduction to these diagrams (for details, see the Sections 3 and 4) and the results for homology.

Let $\mathcal{C}^{(k)}$ be the \mathbb{Z} -module freely generated by the Schubert varieties \mathcal{S}_w , $w \in \mathcal{W}^{(k)}$, the main discussion in this work is how to compute the integer coefficients $c(w, w')$ of the boundary map $\partial : \mathcal{C}^{(k)} \rightarrow \mathcal{C}^{(k)}$. If $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$, the coefficient has the formula $c(w, w') = (-1)^x(1 + (-1)^\kappa)$, where κ is the integer defined by $\phi(w) - \phi(w') = \kappa \cdot \beta$, $\phi(u) = \sum_{\gamma \in \Pi_u} \gamma$ and β is the unique root such that $w = s_\beta w'$.

In general, a element $w \in \mathcal{W}^{(k)}$, either of type B, C or D, can be written as a permutation of the form

$$w = (u_k, \dots, u_1, \bar{z}_1, \dots, \bar{z}_r, v_{n-k-r}, \dots, v_1)$$

where $z_1 > \dots > z_r$, $u_k < \dots < u_1$ and $v_{n-k-r} < \dots < v_1$.

Using the definition in Pragacz-Ratajski [16, 17], the set $\mathcal{W}^{(k)}$ is bijective to set of double partitions $\Lambda = (\alpha|\lambda)$ and each double partition can be represented as a “half” shifted Young diagram (which for groups of type B and C, means that the Young diagram of α fits inside a $k \times (n - k)$ square and λ fits inside a $n \times n$ staircase shape). The benefit to use the half shifted form for the Young diagrams is demonstrated when we need to compute the set Π_w of all positive roots sent

to negative by w^{-1} , which give us a easy way to get the coefficient $\kappa(w, w')$ only drawing its half shifted Young diagrams.

Given a Schubert variety $\mathcal{S}_{w'}$ with partition $\Lambda' = (\alpha'|\lambda')$ such that $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$, the diagram associated to w' is obtained by removing one box from the diagram of w . The partition w' is called

- (1) a α -removing of w if $\alpha' = (\alpha_1, \dots, \alpha_t - 1, \dots, \alpha_k)$ for some $1 \leq t \leq k$;
- (2) a λ -removing of w if $\lambda' = (\lambda_1, \dots, \lambda_t - 1, \dots, \lambda_r)$ for some $1 \leq t \leq r$.

Our main theorem is stated below:

Theorem. *Given w, w' in $\mathcal{W}^{(k)}$ such that $\ell(w) = \ell(w') - 1$, this can be either a α -removing or a λ -removing of w and consider $t = t(w, w')$ the row where the box is removed. The coefficient $c(w, w')$ in the boundary map is obtained from $\kappa(w, w') = t + A(t)$, which depends on the group G and which box is removed:*

$$\begin{aligned} \textbf{Type C: } A(t) &= \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ k & , \text{ for a } \lambda\text{-removing and } \lambda_t = 1; \\ 2k - p + 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = u_p \text{ for some } p; \\ k + n - q + 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = v_q \text{ for some } q. \end{cases} \\ \textbf{Type B: } A(t) &= \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ 2k + t - 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t = 1; \\ 2k - p & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = u_p \text{ for some } p; \\ k + n - q & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = v_q \text{ for some } q. \end{cases} \\ \textbf{Type D: } A(t) &= \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ 2k - p - 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t = u_p \text{ for some } p; \\ k + n - q & , \text{ for a } \lambda\text{-removing and } \lambda_t = v_q \text{ for some } q. \end{cases} \end{aligned}$$

The article is organized as follows. Section 1 presents the preliminaries about the flag manifolds and the Bruhat decomposition. Section 2 summarizes the basic results about the cellular homology of real flag manifolds as obtained in [18]. Section 3 shows the description of the double Young's diagram of all Grassmannians and how it is possible to write the Weyl element from the diagram. Section 4 presents the formulas to get the set Π_w of all positive roots sent to negative by w^{-1} . Section 5 states the main results describing the boundary maps of the cellular homology of Grassmannians of type B, C and D. Section 6 explicitly presents the computations.

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1. PRELIMINARIES

1.1. Flag Manifolds. The main facts about semi-simple Lie Groups and their flag manifolds may be found in Helgason [11], Knapp [13], Warner [22] and San Martin [20]. Furthermore, [15] is the reference for the notation and for some specific results in the article. Flag manifolds are defined as homogeneous spaces G/P where G is a non-compact semi-simple Lie group and P is a parabolic subgroup of G . Let \mathfrak{g} be a non-compact real semi-simple Lie algebra. The flag manifolds for the several groups G with Lie algebra \mathfrak{g} are the same.

Take a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and let \mathfrak{a} be a maximal abelian subalgebra contained in \mathfrak{s} . We denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and fix a simple system of roots $\Sigma \subset \Pi$. Denote by Π^\pm the set of positive and negative roots

respectively and by \mathfrak{a}^+ the Weyl chamber $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\}$. Let $\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ be the direct sum of root spaces corresponding to the positive roots. The Iwasawa decomposition of \mathfrak{g} is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The notations K and N are used to indicate the connected subgroups whose Lie algebras are \mathfrak{k} and \mathfrak{n} respectively.

A sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be a Cartan sub-algebra if $\mathfrak{h}_\mathbb{C}$ is a Cartan sub-algebra of $\mathfrak{g}_\mathbb{C}$. If $\mathfrak{h} = \mathfrak{a}$ is a Cartan sub-algebra of \mathfrak{g} we say that \mathfrak{g} is a split real form of $\mathfrak{g}_\mathbb{C}$.

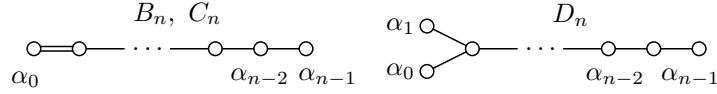
A minimal parabolic sub-algebra of \mathfrak{g} is given by $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where \mathfrak{m} is the centralizer of \mathfrak{a} in \mathfrak{k} . Let P be the minimal parabolic subgroup with Lie algebra \mathfrak{p} . Note that P is the normalizer of \mathfrak{p} in G . We call $\mathbb{F} = G/P$ the maximal flag manifold of G and denote by b_0 the base point $1 \cdot P$ in G/P .

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups. We write $\mathfrak{g}(\Theta)$ for the semi-simple Lie algebra generated by $\mathfrak{g}_{\pm\alpha}$, $\alpha \in \Theta$. Let $G(\Theta)$ be the connected group with Lie algebra $\mathfrak{g}(\Theta)$. Moreover, let \mathfrak{n}_Θ be the sub-algebra generated by the roots spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \Theta$ and put $\mathfrak{p}_\Theta = \mathfrak{n}_\Theta \oplus \mathfrak{p}$.

The normalizer P_Θ of \mathfrak{p}_Θ in G is a standard parabolic subgroup which contains P . The corresponding flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is called a partial flag manifold of G or flag manifold of type Θ . We denote by b_Θ the base point $1 \cdot P_\Theta$ in G/P_Θ .

In this paper the classical split real Lie groups considered are the Orthogonal groups $G = \text{SO}(n, n+1)$, $n \geq 3$, $G = \text{SO}(n, n)$, $n \geq 4$, and the Symplectic Group $G = \text{Sp}(n, \mathbb{R})$, $n \geq 2$, where \mathfrak{g} is the respective Lie algebra given by $\mathfrak{so}(n, n+1)$, $\mathfrak{so}(n, n)$ and $\mathfrak{sp}(n, \mathbb{R})$. According to [20], $\mathfrak{so}(n, n+1)$, $\mathfrak{so}(n, n)$ are real forms of type BDI and $\mathfrak{sp}(n, \mathbb{R})$ of type CI. Given $k = 0, \dots, n-1$, we will work with the minimal flag manifolds of the form G/P_Θ where $\Theta = \Sigma \setminus \{\alpha_k\}$ is a maximal proper subset of the simple set of roots.

Let $\Sigma = \{\alpha_0, \dots, \alpha_{n-1}\}$ be the set of simple roots of \mathfrak{g} ordered as below:



For each choice of group G , the corresponding flag manifold has a geometric realization. The Orthogonal Grassmannian of type B is the set of $(n-k)$ -dimensional isotropic subspaces in the vector space $V = \mathbb{R}^{2n+1}$ equipped with a nondegenerate symmetric bilinear form. It will be denoted by $\text{OG}(n-k, 2n+1) = \mathbb{F}_{\Sigma \setminus \{\alpha_k\}}$.

The Isotropic Grassmannian of type C is the set of $(n-k)$ -dimensional isotropic subspaces in the symplectic vector space $V = \mathbb{R}^{2n}$. This set will be denoted by $\text{IG}(n-k, 2n) = \mathbb{F}_{\Sigma \setminus \{\alpha_k\}}$.

The Orthogonal Grassmannian of type D is the set of $(n+1-k)$ -dimensional isotropic subspaces in the vector space $V = \mathbb{R}^{2n+2}$ equipped with a nondegenerate symmetric bilinear form. It will be denoted by $\text{OG}(n+1-k, 2n+2) = \mathbb{F}_{\Sigma \setminus \{\alpha_k\}}$.

1.2. Bruhat Decomposition. A central role in our context will be played by the Weyl group \mathcal{W} associated to \mathfrak{a} . This is the finite group generated by the reflections over the root hyperplanes $\alpha = 0$ contained in \mathfrak{a} , $\alpha \in \Sigma$. Alternatively, it may be given as the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ where $N_K(\mathfrak{a})$ and $M = Z_K(\mathfrak{a})$ are respectively the normalizer and the centralizer of \mathfrak{a} in K (the Lie algebra of M is \mathfrak{m}).

Viewing the elements of \mathcal{W} as product of simple reflections $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$, the length $\ell(w)$ of $w \in \mathcal{W}$ is the number of simple reflections in any reduced decomposition of w which is equal to the cardinality of $\Pi_w = \Pi^+ \cap w\Pi^-$, the set of positive roots sent to negative roots by w^{-1} . If $w = s_{j_1} \cdots s_{j_r}$ is a reduced decomposition of w then we put

$$(1) \quad \beta_t = s_{j_1} \cdots s_{j_{t-1}}(\alpha_t), \text{ for } 1 \leq t \leq r.$$

It follows that $\Pi_w = \{\beta_1, \dots, \beta_r\}$. This is called the β -sequence of w . There is a partial order in the Weyl group called the Bruhat-Chevalley order. We say that $w_1 \leq w_2$ if given a reduced decomposition $w_2 = s_{j_1} \cdots s_{j_l}$ then $w_1 = s_{j_{i_1}} \cdots s_{j_{i_l}}$ for some $i_1 \leq \dots \leq i_l$.

For a subset $\Theta \subset \Sigma$, the subgroup \mathcal{W}_Θ is defined to be the stabilizer of $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta\}$. Alternatively, \mathcal{W}_Θ may be seen as the subgroup of the Weyl group generated by the reflections with respect to the roots $\alpha \in \Theta$.

We also define the subset \mathcal{W}^Θ of \mathcal{W} by

$$\mathcal{W}^\Theta = \{w \in \mathcal{W} : \ell(ws_\alpha) = \ell(w) + 1, \alpha \in \Theta\}.$$

Since there exists a unique element $w^\Theta \in \mathcal{W}^\Theta$ of minimal length in each coset $w\mathcal{W}_\Theta$, \mathcal{W}^Θ is called the subset of minimal representatives of the cosets of \mathcal{W}_Θ in \mathcal{W} .

The Bruhat decomposition presents the flag manifolds as a union of N -orbits, namely,

$$\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}/\mathcal{W}_\Theta} N \cdot wb_\Theta$$

where $N \cdot w_1 b_\Theta = N \cdot w_2 b_\Theta$ if $w_1 \mathcal{W}_\Theta = w_2 \mathcal{W}_\Theta$.

Each N -orbit through w is diffeomorphic to a euclidean space. Such an orbit $N \cdot wb_\Theta$ is called a Bruhat cell. Its dimension is given by the formula

$$\dim(N \cdot wb_\Theta) = \sum_{\alpha \in \Pi_w \setminus \langle \Theta \rangle} m_\alpha$$

where $m_\alpha = \dim(\mathfrak{g}_\alpha)$ is the multiplicity of the root space \mathfrak{g}_α and $\langle \Theta \rangle$ denotes the roots in Π^+ generated by Θ .

Remark 1.1. Given any $w \in \mathcal{W}/\mathcal{W}_\Theta$, in order to establish a relation between the dimension $\dim(N \cdot wb_\Theta)$ and the length $\ell(w)$, we must choose the minimal representative for w in \mathcal{W}^Θ . In this case, for $w = s_{j_1} \cdots s_{j_r} \in \mathcal{W}^\Theta$, $\dim(N \cdot wb_\Theta) = \sum_{i=1}^r m_{\alpha_{j_i}} + m_{2\alpha_{j_i}}$ (see [23], Corollary 2.6). In the case of a split real form, i.e., $m_{\alpha_{j_i}} = 1$ and $m_{2\alpha_{j_i}} = 0$, it follows that $\dim(N \cdot wb_\Theta) = \ell(w)$.

A Schubert variety is the closure of a Bruhat cell, i.e.,

$$\mathcal{S}_w^\Theta = \text{cl}(N \cdot wb_\Theta).$$

Remark 1.2. For a maximal flag manifold we avoid the superscript Θ and write

$$\mathcal{S}_w = \text{cl}(N \cdot wb_0).$$

The Bruhat-Chevalley order defines an order between the Schubert varieties by

$$\mathcal{S}_{w_1}^\Theta \subset \mathcal{S}_{w_2}^\Theta \text{ if, and only if, } w_1 \leq w_2.$$

A well known characterization of the Schubert varieties is given by

$$\mathcal{S}_w^\Theta = \bigcup_{u \leq w} N \cdot ub_\Theta.$$

I.e., Schubert varieties endow the flag manifolds with a cellular decomposition.

2. CELLULAR HOMOLOGY

We summarize the main results of [18]. For our purpose, we are supposing that G is a group of type B, C or D. Consider firstly a maximal flag manifold \mathbb{F} . The cellular homology of a CW complex is defined from a cellular decomposition of the complex provided in our context by the Schubert varieties. Given a Schubert variety \mathcal{S}_w , we fix once and for all reduced decompositions $w = s_{j_1} \cdots s_{j_r}$ as a product of simple reflections, for each $w \in \mathcal{W}$, and $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$.

Let \mathcal{C} be the \mathbb{Z} -module freely generated by \mathcal{S}_w , $w \in \mathcal{W}$. The boundary maps $\partial : \mathcal{C} \rightarrow \mathcal{C}$ are defined by

$$(2) \quad \partial \mathcal{S}_w = \sum_{w'} c(w, w') \mathcal{S}_{w'}$$

for some coefficients $c(w, w') \in \mathbb{Z}$. In case $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} \neq 1$ then $c(w, w') = 0$. If $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$ then $c(w, w')$ is computed as the degree of a map between cells of codimension one (see [10], page 140).

Proposition 2.1 ([18], Proposition 4.1). *Let $w, w' \in \mathcal{W}$. The following statements are equivalent.*

- (1) $\mathcal{S}_{w'} \subset \mathcal{S}_w$ and $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$;
- (2) If $w = s_{j_1} \cdots s_{j_r}$ is a reduced decomposition of $w \in \mathcal{W}$ as a product of simple reflections, then $w' = s_{j_1} \cdots \hat{s}_{j_i} \cdots s_{j_r}$ is a reduced decomposition and $\mathfrak{g}(\alpha_i) \cong \mathfrak{sl}(2, \mathbb{R})$.

The first main result is that the coefficient $c(w, w')$ is the sum of the degree of two sphere homeomorphisms which has degree one.

Theorem 2.2 ([18], Theorem 4.3). *For any choice $w, w' \in \mathcal{W}$ such that $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$, the coefficient $c(w, w') = 0, \pm 2$.*

It is possible to get a more accurate expression for the coefficients $c(w, w')$ in terms of roots. For $w = s_{j_1} \cdots s_{j_r}$ and $w' = s_{j_1} \cdots \hat{s}_{j_i} \cdots s_{j_r}$, reduced decompositions of w and w' respectively (see Proposition 2.1), we define $\chi = \chi(w, w') = i$ and

$$(3) \quad \sigma = \sigma(w, w') = \sum_{\beta \in \Pi_u} \frac{2\langle \alpha_i, \beta \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad u = s_{j_{i+1}} \cdots s_{j_r}.$$

Theorem 2.3 ([18], Theorem 4.7). *Let be $\sigma(w, w')$ be defined as in (3). Then*

$$c(w, w') = (-1)^x (1 - (-1)^\sigma).$$

For $w \in \mathcal{W}$, let

$$(4) \quad \phi(w) = \sum_{\beta \in \Pi_w} \beta$$

be the sum of roots in $\Pi_w = \Pi^+ \cap w\Pi^-$, i.e., the sum of the β -sequences of w .

Given w and w' such that $\ell(w) = \ell(w') + 1$, let β be the unique root (not necessarily simple) satisfying $w = s_\beta w'$, that is, $\beta = s_1 \cdots s_{i-1} \alpha_i$. Then it is possible to show that

$$\phi(w) - \phi(w') = (1 - \sigma)\beta$$

where σ is the sum (3) (see [18], Proposition 4.8). We get immediately the following formula for $c(w, w')$.

Theorem 2.4 ([14], Theorem A and [18], Theorem 4.9).

$$(5) \quad c(w, w') = (-1)^x (1 + (-1)^\kappa)$$

where $\kappa = \kappa(w, w')$ is the integer defined by $\phi(w) - \phi(w') = \kappa \cdot \beta$ and β is the unique root such that $w = s_\beta w'$.

Remark 2.5. *This result was first obtained by Kocherlakota (see Theorem A, [14]) by a Morse homology approach. The advantage here is working with reduced decompositions to compute the numbers $\phi(w)$.*

In the context of the partial flag manifolds \mathbb{F}_Θ , the Schubert varieties are \mathcal{S}_w^Θ , $w \in \mathcal{W}^\Theta$.

Theorem 2.6 ([18], Theorem 5.4). *The cellular homology of \mathbb{F}_Θ is isomorphic to the homology of ∂_{\min}^Θ which is the boundary map of the free module A_{\min}^Θ generated by \mathcal{S}_w , $w \in \mathcal{W}^\Theta$, obtained by restricting ∂ and projecting onto A_{\min}^Θ .*

Hence the coefficients $c^\Theta([w], [w'])$ for the boundary map ∂^Θ of the cellular homology of the partial flag manifolds \mathbb{F}_Θ is

$$c^\Theta([w], [w']) = c(w, w').$$

and the computation of $c^\Theta([w], [w'])$ reduces to a computation of $c(w, w')$ on \mathbb{F} .

3. HALF-SHIFTED YOUNG DIAGRAMS

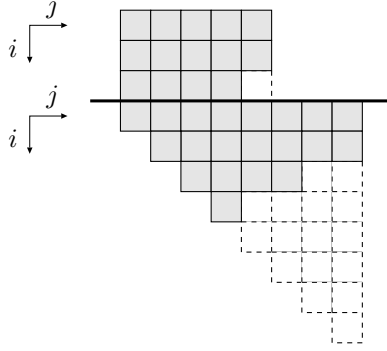
The goal of this section is to provide the main tool for dealing with the combinatorics of isotropic Grassmannians. The complexity of the Weyl group that parametrizes the cellular decomposition of these manifolds has a very nice description in terms of what we call *half shifted* Young diagrams. The main source to construct such diagrams is the work of Buch-Kresch-Tamvakis [3, 4], where the model of k -strict partition and Young diagrams is presented together with properties of Grassmannians of all types. Another reference is given by Ikeda-Naruse [12] and also by Graham-Kreiner [9] in the context of maximal Grassmannians where its found how to fill in each Young diagram with the positive roots sent to negative by the respective Weyl group element. Here we propose a model where it is generalized by showing how to fill in the diagrams defined by BKT for all types of Grassmannians with the positive roots sent to negative by the respective Weyl group element. As far as we know, such a description has not been done yet elsewhere. In particular, this will be very useful to compute boundary maps for cellular homology.

3.1. The isotropic Grassmannian $\text{IG}(n - k, 2n)$. Consider the Grassmannian $\text{IG}(n - k, 2n)$ which parametrizes $(n - k)$ -dimensional isotropic subspaces of a real $2n$ -dimensional symplectic vector space.

Let $\mathcal{R}(m, n)$ denote the set of integer partitions $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m \geq 0)$ with $\alpha_1 \leq n$ so that the Young diagrams of α fits inside a $m \times n$ rectangle. Also define \mathcal{D}_n the set of all strict partitions $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_l > 0)$ with $\lambda_1 \leq n$.

The Weyl group \mathcal{W}_n for the root system C_n are permutations in S_n with a sign attached to each entry, that means it is isomorphic to the semidirect product $S_n \rtimes \mathbb{Z}_2^n$; we will write these elements as barred permutations of the form

$$\bar{n}, \dots, \bar{2}, \bar{1}, 1, 2, \dots, n$$

FIGURE 1. Half shifted Schubert cell $(5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$.

using the bar to denote a negative sign, and we take the natural order on them, as above. If \mathcal{W}_k is the parabolic subgroup generated by $\{s_i : i \neq k\}$ then the set $\mathcal{W}^{(k)} \subset \mathcal{W}_n$ of minimal length coset representatives of \mathcal{W}_k parametrizes the Schubert varieties in $\text{IG}(n-k, 2n)$. This indexing set $\mathcal{W}^{(k)}$ can be identified by a set of barred permutations of the form

$$(6) \quad w = w_{u,\lambda} = (u_k, \dots, u_1, \bar{\lambda}_1, \dots, \bar{\lambda}_r, v_{n-k-r}, \dots, v_1)$$

where $\lambda \in \mathcal{D}_n$ with $r = \ell(\lambda) \leq n-k$, $0 < u_k < \dots < u_1$ and $0 < v_{n-k-r} < \dots < v_1$.

There is a description of elements of $\mathcal{W}^{(k)}$ by means of Young diagrams, i.e., each element $w_{u,\lambda} \in \mathcal{W}^{(k)}$ corresponds to a pair of partitions $\Lambda = (\alpha|\lambda)$ where the “top” partition $\alpha = \alpha(u, \lambda)$ is defined slightly different from Pragacz-Ratajski [16]

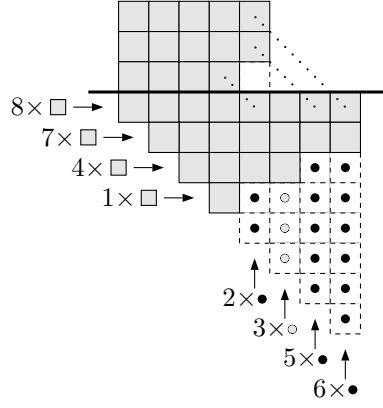
$$(7) \quad \alpha_i = u_i + i - k - 1 + d_i$$

for $1 \leq i \leq k$ and $d_i = \#\{j \mid \lambda_j > u_i\}$; the “bottom” partition is λ . Note that the length of w is $\ell = \ell(w) = |\alpha| + |\lambda|$, where $|\alpha| := \sum_{j=1}^k \alpha_j$ and $|\lambda| := \sum_{j=1}^r \lambda_j$.

The Schubert varieties \mathcal{S}_Λ are then parametrized by the set $\mathcal{P}(k, n)$ of pairs $\Lambda = (\alpha|\lambda)$ with $\alpha \in \mathcal{R}(k, n-k)$, $\lambda \in \mathcal{D}_n$ and such that $\alpha_k \geq \ell(\lambda)$. We may arrange the top and the bottom partition in the same diagram. We propose here an alternative way of arrange the diagram of Λ which corresponds just to a “half” shifted diagram in the sense presented by Tamvakis [21]. We consider the top diagram α left justified in the $k \times (n-k)$ rectangle and, for each $1 \leq i \leq r$, the i -th row of λ is shifted to the right $i-1$ units. With that shift, the bottom diagram may be seen inside a staircase partition and for reasons that will be clear soon we consider the staircase partition with n rows. The condition $\alpha_k \geq \ell(\lambda)$ implies that the number of rows in the bottom diagram does not exceed the number of boxes in the last row of the top diagram.

Let us denote by D_α and by D'_λ the set of square boxes with coordinates $(i, j) \in \mathbb{Z}^2$ the top and the bottom (shifted) diagrams, respectively. Each diagram D_α and D'_λ is arranged in a plane with matrix-style coordinates. The Figure 1 presents the half shifted diagram of the Schubert variety $(5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$ and how to read a (i, j) box in each diagram.

For each row in the top diagram, construct a diagonal line (indicated as a dashed line in the Figure 2) from the center of the last box at right of the respective row to the center of the first box in the first row of the bottom diagram (it means


 FIGURE 2. Computing permutation of $(5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$.

that these boxes are k -related in this half shifted diagram as defined by Tamvakis [21]). Consider the boxes of the staircase partition that are outside λ organized into columns. The columns that contain a such k -related box are called *related* and the remaining columns are *non-related*. The *length* of a column is the number of blank boxes of this column in the staircase $n \times n$ shape. We may recover the permutation associated to such diagram by taking the length of the related and non-related columns.

Namely, the permutation element for $\Lambda = (\alpha, \lambda)$ is defined by w_Λ in the Equation (6), where $0 < u_k < \dots < u_1$ are the length of the related columns, $r = \ell(\lambda)$ and $0 < v_{n-k-r} < \dots < v_1$ are the length of the non-related columns. For example, the partition $\Lambda = (5, 5, 4|8, 7, 4, 1)$ corresponds to the element $w_\Lambda = (2, 5, 6, 8, 7, 4, 1, 3)$ as illustrated in the Figure 2.

Remark 3.1. *This description represents a slight modification with respect to that is given in [3] in terms of the model of k -strict partitions. The choice for representing Schubert varieties by half shifted diagrams instead of a diagram associated to a k -strict partition will be clarified in the next section.*

We now give a notion of row-reading expression for each $w_\Lambda \in \mathcal{W}^{(k)}$, $\Lambda = (\alpha, \lambda) \in \mathcal{P}(k, n)$. With respect to the top partition α , let $s^T : D_\alpha \rightarrow \{s_1, \dots, s_{n-1}\}$ be defined, for $1 \leq i \leq k$ and $1 \leq j \leq n - k$, by

$$(8) \quad s^T(i, j) = s_{j-i+k}.$$

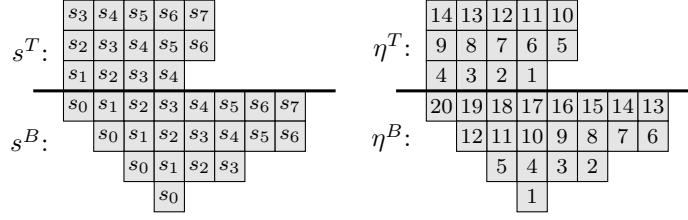
For a given partition $\alpha \in \mathcal{R}(k, n - k)$ associated to $\Lambda = (\alpha, \lambda) \in \mathcal{P}(k, n)$, the row-reading map is a bijection $\eta^T : D_\alpha \rightarrow \{1, 2, \dots, |\alpha|\}$ defined by assign the numbers increasingly to the boxes of D_α from right to left starting from the bottom row to the top row. Then, we can form a word

$$(9) \quad w_\alpha = s_{i_1} \cdots s_{i_{|\alpha|}}$$

where $s_{i_l} = s^T((\eta^T)^{-1}(l))$, for all $1 \leq l \leq |\alpha|$.

With respect to the bottom partition λ , let $s^B : D'_\lambda \rightarrow \{s_0, \dots, s_{n-1}\}$, for $1 \leq i \leq n - k$, $1 \leq j \leq n$ and $i \leq j$, by

$$(10) \quad s^B(i, j) = s_{j-i}.$$

FIGURE 3. Row-reading of $(5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$.

As before, for a given partition $\lambda \in \mathcal{D}_n$ associated to $\Lambda = (\alpha, \lambda) \in \mathcal{P}(k, n)$, the row-reading map is a bijection $\eta^B : D'_\lambda \rightarrow \{1, 2, \dots, |\lambda|\}$ defined by assign the numbers increasingly to the boxes of D_α from right to left starting from the bottom row to the top row. Then, we can form a word

$$(11) \quad w_\lambda = s_{j_1} \cdots s_{j_{|\lambda|}}$$

where $s_{j_l} = s^B((\eta^B)^{-1}(l))$, for all $1 \leq l \leq |\lambda|$. The concatenation of expressions (11) and (9) gives $w = w_\Lambda = w_\lambda w_\alpha$.

The maps s^T , s^B , η^T and η^B for the Schubert variety given by the partition $(5, 5, 4|8, 7, 4, 1)$ of $\text{IG}(5, 16)$ are illustrated in Figure 3. Then, $w_\lambda = s_0 \cdot s_3 s_2 s_1 s_0 \cdot s_6 s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_0$ and $w_\alpha = s_4 s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2 \cdot s_7 s_6 s_5 s_4 s_3$.

3.2. The odd orthogonal Grassmannian $\text{OG}(n - k, 2n + 1)$. Consider the odd orthogonal Grassmannian $\text{OG}(n - k, 2n + 1)$ which parametrizes $(n - k)$ -dimensional isotropic subspaces of a real $(2n + 1)$ -dimensional vector space equipped with a nondegenerate symmetric bilinear form. The Weyl group for the root system B_n is the same as that for C_n . Hence, they have the same diagrams. The only distinction will appear in the level of roots. This will be described in the next section.

3.3. The even orthogonal Grassmannian $\text{OG}(n + 1 - k, 2n + 2)$. Consider the Grassmannian $\text{OG}' = \text{OG}(n + 1 - k, 2n + 2)$ which parametrizes $(n + 1 - k)$ -dimensional isotropic subspaces of a real $(2n + 2)$ -dimensional vector space equipped with a nondegenerate symmetric form. We denote by $\widetilde{\mathcal{W}}_{n+1}$ the Weyl group of type D_{n+1} . The elements of $\widetilde{\mathcal{W}}_{n+1}$ are signed permutations of $\{1, 2, \dots, n\}$ with an even number of sign changes which will be denoted by barred numbers.

If $\widetilde{\mathcal{W}}_k$ is the subgroup of $\widetilde{\mathcal{W}}_{n+1}$ generated by $\{s_i \mid i \neq k\}$, then the set $\widetilde{\mathcal{W}}^{(k)} \subset \widetilde{\mathcal{W}}_{n+1}$ of minimal length coset representatives of $\widetilde{\mathcal{W}}_k$ parametrizes the Schubert varieties in OG' . The set $\widetilde{\mathcal{W}}^{(k)}$ consists of barred permutations of the form

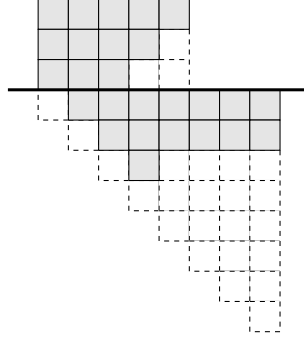
$$(12) \quad w = w_{u, \lambda} = (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_1)$$

where $\lambda_1 > \dots > \lambda_r > 0$, $0 < u_k < \dots < u_1$, $v_{n-k-r} < \dots < v_1$, $\bar{1} \leq v_{n+1-k-r} < v_{n-k-r}$ and \hat{u}_k is equal to u_k or \bar{u}_k so that the expression has an even number of bars.

The elements in $\widetilde{\mathcal{W}}^{(k)}$ correspond to a set $\widetilde{\mathcal{P}}(k, n)$ of partition pairs, which involve a partition $\alpha \in \mathcal{R}(k, n + 1 - k)$ and a $\lambda \in \mathcal{D}_n$ such that $\alpha_k \geq \ell(\lambda)$. More precisely, define the partition α as

$$\alpha_i = u_i + i - k - 1 + \#\{j \mid \lambda_j + 1 > u_i\}$$

for $1 \leq i \leq k$. Then, $\widetilde{\mathcal{P}}(k, n)$ consists of


 FIGURE 4. The half shifted diagram $(5, 4, 3|7, 6, 1)$ in $\text{OG}(5, 16)$.

- (i) pairs $(\alpha|\lambda)$ such that $\alpha_k = \ell(\lambda)$; these are elements of $\widetilde{\mathcal{W}}^{(k)}$ of the form

$$w_{(\alpha|\lambda)} = (\widehat{1}, u_{k-1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_1)$$

which is referred as *of type 0*;

- (ii) two types of pairs $[\alpha|\lambda]$ and $(\alpha|\lambda)$ such that $\alpha_k > \ell(\lambda)$; these correspond to elements of $\widetilde{\mathcal{W}}^{(k)}$ of the form

$$w_{[\alpha|\lambda]} = (\widehat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, 1, v_{n-k-r}, \dots, v_1)$$

which is referred as *of type 1* and

$$w_{(\alpha|\lambda]} = (\widehat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, \overline{1}, v_{n-k-r}, \dots, v_1)$$

which is referred as *of type 2*.

With such description we may represent these elements by means of Young diagrams, i.e., the pair of partitions $\Lambda = [\alpha|\lambda]$ (where $[\alpha, \lambda]$ means a generic partition of any type) provides the “top” partition α and the “bottom” partition is λ . The Schubert varieties \mathcal{S}_Λ are then parametrized by the set $\widetilde{\mathcal{P}}(k, n)$ of pairs $\Lambda = [\alpha|\lambda]$ with $\alpha \in \mathcal{R}(k, n+1-k)$, $\lambda \in \mathcal{D}_n$ and such that $\alpha_k \geq \ell(\lambda)$.

We also propose likewise in the type C case an alternative way of arrange the diagram of Λ which corresponds just to a “half” shifted diagram in the sense presented by Tamvakis (see [21], page 19). We consider the top diagram α left justified in the $k \times (n+1-k)$ rectangle and, for each $1 \leq i \leq r$, the i -th row of λ is shifted to the right i units. With that shift, the bottom diagram may be seen inside a staircase partition with $n+1$ rows.

Also denote by D_α and by D'_λ the set of square boxes with coordinates $(i, j) \in \mathbb{Z}^2$ the top and the bottom (shifted) diagrams, respectively. Each diagram D_α and D'_λ is arranged in a plane with matrix-style coordinates. For example, consider the Schubert variety $\alpha = (5, 4, 3)$ and $\lambda = (7, 6, 1)$ in $\text{OG}(5, 16)$. Notice that $\alpha_3 = 3 = \ell(\lambda)$ which means that it is of type 0. The diagram and its respective half shifted diagram are illustrated in Figure 4.

For each row in the top diagram, construct a diagonal line (indicated as a dashed line in the Figure 5) from the center of the last box at right of the respective row to the center of the first box in the first row of the bottom diagram (it means that these boxes are k -related in this half shifted diagram as defined by Tamvakis [21]). Consider the boxes of the staircase partition that are outside λ organized into columns. The columns that contain a such k -related box are called *related*

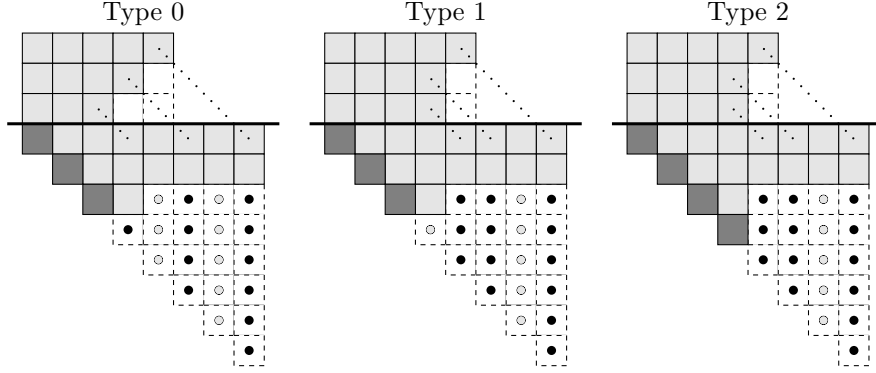


FIGURE 5. Permutation of $(5, 4, 3|7, 6, 3)$ (left), $[5, 4, 4|7, 6, 3]$ (center) and $(5, 4, 4|7, 6, 3)$ (right) in $\text{OG}(5, 16)$.

and the remaining columns are *non-related*. The *length* of a column is the number of blank boxes of this column in the staircase $n \times n$ shape. We may recover the permutation associated to such diagram by taking the length of the related and non-related columns.

Namely, the permutation element for $\Lambda = [\alpha, \lambda]$ is defined by w_Λ in the Equation (12), where $0 < u_k < \dots < u_1$ are the length of the related columns, $v_{n+1-k-p} < \dots < v_1$ are the length of the non-related columns. For example, the Schubert variety $\Lambda = (5, 4, 3|7, 6, 1)$ of type 0 in $\text{OG}(5, 16)$ corresponds to the permutation $w_\Lambda = (\bar{1}, 4, 6, \bar{8}, \bar{7}, \bar{2}, 3, 5)$ as illustrated in Figure 5 (left).

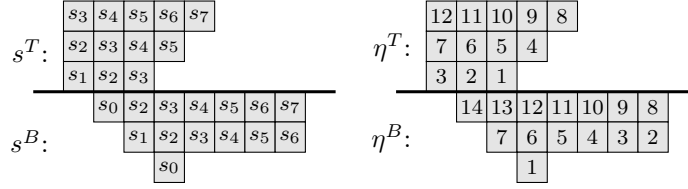
Notice that one must fill in the diagonal boxes of the first $\ell(\lambda)$ rows that appear in the shifting procedure. If the diagram is of type 0, we have a unique diagram since $\alpha_k = \ell(\lambda)$. If the diagram is of type 1 or 2, since $\ell(\lambda) < \alpha_k$, we will have two possible diagrams: one of them with the diagonal box in the $(\ell(\lambda) + 1)$ -row not filled in and another with such box filled in. By definition, they will correspond to the elements of type 1 and of type 2, respectively. Consider, for example, the Schubert varieties given by $\Lambda_1 = [5, 4, 4|7, 6, 1]$ and $\Lambda_2 = (5, 4, 4|7, 6, 1)$ of type 1 and 2, respectively, inside $\text{OG}(5, 16)$ as illustrated in Figure 5. The permutations are, respectively, $w_{\Lambda_1} = (\bar{3}, 4, 6, \bar{8}, \bar{7}, \bar{2}, 1, 5)$ and $w_{\Lambda_2} = (3, 4, 6, \bar{8}, \bar{7}, \bar{2}, \bar{1}, 5)$.

Remark 3.2. *The diagrams presented here of type D are also slight modifications of the model of the so called typed k-strict partitions presented by Buck-Kresch-Tamvakis. For details, see [4], Section 6. We notice that the diagrams presented here have the advantage of incorporating the +1 factor that appears in the BKT-model. Finally, we hope that the distinction between the type 1 and 2 Weyl group elements were clarified in terms of its diagrams.*

As it was made for groups of type C and B we have a row-reading expression for each $w_\Lambda \in \widetilde{\mathcal{W}}^{(k)}$, $\Lambda = [\alpha, \lambda] \in \mathcal{P}(k, n)$. We start with the maps s^T, s^B that says how the boxes of a diagram are filled in with simple reflections.

Type 0: With respect to the top partition α , let $s^T : D_\alpha \rightarrow \{s_0, \dots, s_n\}$ be defined, for $1 \leq i \leq k$ and $1 \leq j \leq n - k$, by

$$(13) \quad s^T(i, j) = s_{j-i+k}.$$


 FIGURE 6. Row-reading of $(5, 4, 3|7, 6, 1)$ in $\text{OG}(5, 16)$.

With respect to the bottom partition λ , let $s^B : D'_\lambda \rightarrow \{s_0, \dots, s_n\}$ be defined, for $1 \leq i \leq n - k$, $1 \leq j \leq n$ and $i \leq j$, by

$$(14) \quad s^B(i, j) = \begin{cases} s_{j-i} & , \text{ if } i + 1 < j; \\ s_1 & , \text{ if } i + 1 = j \text{ and } i \text{ is even;} \\ s_0 & , \text{ if } i + 1 = j \text{ and } i \text{ is odd,} \end{cases}$$

Types 1 and 2: The case of types 1 and 2 also depends on the parity of the length of λ . With respect to the top partition α , $s^T : D_\alpha \rightarrow \{s_0, \dots, s_n\}$ is defined, for $1 \leq i \leq k$ and $1 \leq j \leq n - k$, by

$$(15) \quad s^T(i, j) = \begin{cases} s_{j-i+k} & , \text{ if either } i \neq k \text{ or } j \neq 1; \\ s_1 & , \text{ if } i = k, j = 1 \text{ and } (\ell(\lambda) + \text{type}(w)) \text{ is odd;} \\ s_0 & , \text{ if } i = k, j = 1 \text{ and } (\ell(\lambda) + \text{type}(w)) \text{ is even.} \end{cases}$$

With respect to the bottom partition λ , $s^B : D'_\lambda \rightarrow \{s_0, \dots, s_n\}$ is defined, for $1 \leq i \leq n - k$, $1 \leq j \leq n$ and $i \leq j$, by

$$(16) \quad s^B(i, j) = \begin{cases} s_{j-i} & , \text{ if } i + 1 < j \\ s_1 & , \text{ if } i + 1 = j \text{ and } (i + \ell(\lambda) + \text{type}(w)) \text{ is odd} \\ s_0 & , \text{ if } i + 1 = j \text{ and } (i + \ell(\lambda) + \text{type}(w)) \text{ is even.} \end{cases}$$

Remark 3.3. Observe, in the Equation (16), that for $i = \ell(\lambda)$ and $j = \ell(\lambda) + 1$, i.e., for the first box at the last row of λ , the reflection is s_1 if w has type 1 and s_0 if w has type 2. Then, we can easily find out the type of w if we know which one appears first in the word: s_1 or s_0 , implying it is of type 1 or 2, respectively.

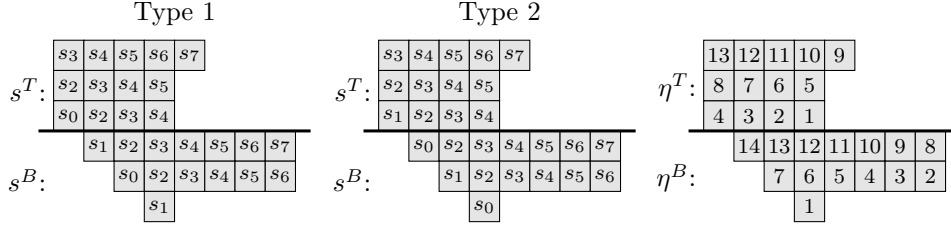
Now, we define the reading row map for all types 0, 1 and 2 in the same way as made for groups of type C: define $\eta^T : D_\alpha \rightarrow \{1, 2, \dots, |\alpha|\}$ and $\eta^B : D'_\lambda \rightarrow \{1, 2, \dots, |\lambda|\}$ by assign the numbers increasingly to the boxes of D_α and D'_λ , respectively, from right to left starting from the bottom row to the top row. Then, we can form the words

$$(17) \quad w_\alpha = s_{i_1} \cdots s_{i_{|\alpha|}}, \quad w_\lambda = s_{j_1} \cdots s_{j_{|\lambda|}}$$

$s_{i_l} = s^T((\eta^T)^{-1}(l))$ and $s_{j_m} = s^B((\eta^B)^{-1}(m))$. The concatenation of the Equations (17) gives $w_\Lambda = w_\lambda w_\alpha$.

The maps s^T , s^B , η^T and η^B for the Schubert variety given by the partition of type 0 $(5, 4, 3|7, 6, 1)$ of $\text{OG}(5, 16)$ are illustrated in the Figure 6.

For the Schubert variety given by the partition of type 1 $[5, 4, 4|7, 6, 1]$ and type 2 $(5, 4, 4|7, 6, 1)$ of $\text{OG}(5, 16)$, this maps are described in the Figure 7.

FIGURE 7. Row-reading of $[5, 4, 4|7, 6, 1]$ and $(5, 4, 4|7, 6, 1]$ in $\text{OG}(5, 16)$.

4. ROOTS AND DIAGRAMS

An important data needed to compute homology of groups is the description of the set $\Pi_w = \Pi^+ \cap w\Pi^-$ of positive roots sent to negative roots by w^{-1} . This section is devoted to show the relationship between $\Pi_w = \Pi^+ \cap w\Pi^-$ and the respective half shifted Young diagram of w . More specifically, since the number of roots in Π_w is equal to length $\ell(w)$, there is a rule that shows how each box in the diagram of w is filled in with a root of Π_w .

4.1. Types B and C. Our purpose is to describe the β -sequences of w using the position (i, j) of a box in the diagram of w .

Let \mathcal{S}_Λ be the Schubert variety parametrized by $\Lambda = (\alpha|\lambda)$ with $\alpha \in \mathcal{R}(k, n-k)$ and $\lambda \in \mathcal{D}_n$. For a box $(i, j) \in D'_\lambda$ of λ define $\beta_{i,j}^B := \beta_{\eta^B(i,j)}^\lambda$, where β_t^λ is the β -sequence for w_λ . For a box $(i, j) \in D_\alpha$ of α define $\beta_{i,j}^T := w_\lambda \cdot \beta_{\eta^T(i,j)}^\alpha$, where β_t^α is the β -sequence for w_α .

Observe that w_λ is a permutation obtained by reordering w according to the order of indexes given by $\bar{n} < \dots < \bar{1} < 1 < \dots < n$. This w_λ also can be seen as the element of the set $\mathcal{W}^{(n)}$ of Lagrangian Grassmannians $\text{IG}(n, 2n)$ related to the shifted diagram λ .

Proposition 4.1. *The β -sequences can be described as follows:*

- The top part will be the same for both types C and B:

$$\beta_{i,j}^T = \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+j)} \quad , \text{ for } (i, j) \in D_\alpha$$

- The bottom part is different for each case:

(1) **Type C:** $\beta_{i,j}^B = \varepsilon_{w_\lambda(2n+1-i)} - \varepsilon_{w_\lambda(j)}$;

(2) **Type B:** $\beta_{i,j}^B = 2^{-\delta_{ij}}(\varepsilon_{w_\lambda(2n+1-i)} - \varepsilon_{w_\lambda(j)})$;

for $(i, j) \in D'_\lambda$

Proof. By definition, β_t^B is the β -sequence for w_λ and this formula is done in [12]. To proof for the “top” part, it is enough to apply [12, Lemma 3] to the definition of $\beta_{i,j}^T$. \square

This proposition give us an easier way to compute the set Π_w of all β -sequences by placing such roots into each box of the Young diagram.

Example 4.2. Consider $\Lambda = (5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$. In this case, $w_\Lambda = (2, 5, 6, \bar{8}, \bar{7}, \bar{4}, \bar{1}, 3)$ and $w_\lambda = (8, \bar{7}, \bar{4}, \bar{1}, 2, 3, 5, 6)$. First of all, we must label the rows and the columns of top diagram using w , starting from the bottom row to the top and, afterwards, from the left column to the right. Then, the β -sequence associated to each box $(i, j) \in D_\alpha$ is the root $\varepsilon_a - \varepsilon_b$, where a and b are the label of

	$\bar{8}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	3		$\bar{8}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	2	3	5	6
6	$\varepsilon_6 + \varepsilon_8$	$\varepsilon_6 + \varepsilon_7$	$\varepsilon_4 + \varepsilon_6$	$\varepsilon_1 + \varepsilon_6$	$\varepsilon_6 - \varepsilon_3$	8	$2\varepsilon_8$	$\varepsilon_7 + \varepsilon_8$	$\varepsilon_4 + \varepsilon_8$	$\varepsilon_1 + \varepsilon_8$	$\varepsilon_8 - \varepsilon_2$	$\varepsilon_8 - \varepsilon_3$	$\varepsilon_8 - \varepsilon_5$	$\varepsilon_8 - \varepsilon_6$
5	$\varepsilon_5 + \varepsilon_8$	$\varepsilon_5 + \varepsilon_7$	$\varepsilon_4 + \varepsilon_5$	$\varepsilon_1 + \varepsilon_5$	$\varepsilon_5 - \varepsilon_3$	7		$2\varepsilon_7$	$\varepsilon_4 + \varepsilon_7$	$\varepsilon_1 + \varepsilon_7$	$\varepsilon_7 - \varepsilon_2$	$\varepsilon_7 - \varepsilon_3$	$\varepsilon_7 - \varepsilon_5$	$\varepsilon_7 - \varepsilon_6$
2	$\varepsilon_2 + \varepsilon_8$	$\varepsilon_2 + \varepsilon_7$	$\varepsilon_2 + \varepsilon_4$	$\varepsilon_1 + \varepsilon_2$		4			$2\varepsilon_4$	$\varepsilon_1 + \varepsilon_4$	$\varepsilon_4 - \varepsilon_2$	$\varepsilon_4 - \varepsilon_3$		
						1				$2\varepsilon_1$				

 FIGURE 8. β -sequences associated to the partition $(5, 5, 4|8, 7, 4, 1)$.

i -th row and j -th column, respectively. For example, the box $(1, 1) \in D_\alpha$ has labels 6 and $\bar{8}$ for row and column, respectively, and it implies that $\beta_{(1,1)}^T = \varepsilon_6 - \varepsilon_{\bar{8}} = \varepsilon_6 + \varepsilon_8$ as we can see in the Figure 8.

For the bottom diagram, we must label the rows using the reflection of w_λ

$$((w_\lambda)_{n+1}, \dots, (w_\lambda)_{2n}) = (\bar{6}, \bar{5}, \bar{3}, \bar{2}, 1, 4, 7, 8)$$

from the bottom row (the n -th row at $n \times n$ staircase) to the top and using w_λ to label the columns from the left to the right. The β -sequences are obtained likely to the top part. The Figure 8 shows all the β -sequences of the partition Λ .

4.2. Type D. Let \mathcal{S}_Λ the Schubert variety parametrized by pairs $\Lambda = [\alpha|\lambda]$ with $\alpha \in \mathcal{R}(k, n+1-k)$, $\lambda \in \mathcal{D}_n$. For a box $(i, j) \in D'_\lambda$ of λ we define $\beta_{i,j}^B := \beta_{(i,j)}^\lambda$, where β_t^λ is the β -sequence for w_λ . For a box $(i, j) \in D_\alpha$ of α we define $\beta_{i,j}^T := w_\lambda \cdot \beta_{(i,j)}^\alpha$, where β_t^α is the β -sequence for w_α .

We denote by \check{w} the permutation associated to $w_{[\alpha,\lambda]}$, $w_{(\alpha,\lambda]}$ where we remove the hat from the first entry \hat{u}_1 (observe that \check{w} might not lie in $\widetilde{\mathcal{W}}^{(k)}$). Notice that w_λ is a permutation obtained by reordering w according to the order of indexes given by $\overline{n+1} < \dots < \bar{1} < 1 < \dots < n+1$. This w_λ also can be seen as the element of the set $\mathcal{W}^{(n)}$ of the Grassmannian of $(n+1)$ -dimensional isotropic subspaces $\text{OG}(n+1, 2n+2)$ related to the shifted diagram λ . Define \check{w}_λ by reordering \check{w} according to the order of indexes $\overline{n+1} < \dots < \bar{1} < 1 < \dots < n+1$.

Proposition 4.3. *The β -sequences can be described as follows:*

- The top part will be given by:

$$\beta_{i,j}^T = \begin{cases} \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+j)} & , \text{ if } \text{type}(w) = 0 \\ \varepsilon_{\check{w}(k-i+1)} - \varepsilon_{\check{w}(k+j)} & , \text{ if } \text{type}(w) = 1, 2 \end{cases} , \text{ for } (i, j) \in D_\alpha;$$

- The bottom part will be given by:

$$\beta_{i,j}^B = \begin{cases} \varepsilon_{w_\lambda(2n+3-i)} - \varepsilon_{w_\lambda(j)} & , \text{ if } \text{type}(w) = 0 \\ \varepsilon_{\check{w}_\lambda(2n+3-i)} - \varepsilon_{\check{w}_\lambda(j)} & , \text{ if } \text{type}(w) = 1, 2 \end{cases} , \text{ for } (i, j) \in D'_\lambda.$$

Proof. Considering each type separately, this proof is similar to Proposition 4.1. \square

Example 4.4. Consider $\Lambda_1 = [5, 4, 4|7, 6, 1]$ of type 1 and $\Lambda_2 = (5, 4, 4|7, 6, 1]$ of type 2 in $\text{OG}(5, 16)$. In this case, we have $\check{w}_{\Lambda_1} = (3, 4, 6, \bar{8}, \bar{7}, \bar{2}, 1, 5)$ and $\check{w}_{\Lambda_2} = (3, 4, 6, \bar{8}, \bar{7}, \bar{2}, \bar{1}, 5)$; $\check{w}_{\lambda_1} = (\bar{8}, \bar{7}, \bar{2}, 1, 3, 4, 5, 6)$ and $\check{w}_{\lambda_2} = (\bar{8}, \bar{7}, \bar{2}, \bar{1}, 3, 4, 5, 6)$. Repeating the same process as in the Example 4.2 but now using the permutation

Figure 1 shows two Cartan matrices. The left matrix is for E_6 with rows labeled 6, 4, 3 and columns labeled 8, 7, 2, 1, 5. The right matrix is for E_7 with rows labeled 8, 7, 2, 1 and columns labeled 8, 7, 2, 1, 3, 4, 5, 6. Both matrices show the Cartan matrix structure with diagonal elements 2 and off-diagonal elements -1 or -2.

FIGURE 9. β -sequences of the partition $[5, 4, 4|7, 6, 1)$ of type 1.

Figure 1 displays two 6x6 grids representing the 6x6 grid of the 15-puzzle. The left grid shows the initial state, and the right grid shows the goal state. The tiles are labeled with numbers 1-5 and their inverses (1̄, 2̄, 3̄, 4̄, 5̄). The goal state uses numbers 1-6 and their inverses (1̄, 2̄, 3̄, 4̄, 5̄, 6̄). Dashed lines indicate the 3x3 blocks in each grid.

FIGURE 10. β -sequences of the partition $(5, 4, 4|7, 6, 1]$ of type 2.

\tilde{w} to label the rows and columns instead of use the original w , the Figures 9 and 10 show the β -sequences for the partitions Λ_1 and Λ_2 , respectively.

5. MAIN RESULTS

Let G be either an orthogonal or a symplectic group and let P_Θ be the parabolic subgroup, where Θ is the complementary simple roots of $\{\alpha_k\}$ of the respective Dynkin diagram. Let \mathcal{C} be the \mathbb{Z} -module freely generated by the Schubert varieties \mathcal{S}_w , $w \in \mathcal{W}^{(k)}$. The coefficients $c(w, w')$ of the boundary map $\partial : \mathcal{C} \rightarrow \mathcal{C}$ are defined by the Equation (2). The cellular homology of the isotropic Grassmannian $\text{IG}(n-k, 2n)$, odd and even orthogonal Grassmannians $\text{OG}(n-k, 2n+1)$ and $\text{OG}(n-k+1, 2n+2)$ is obtained applying the Theorem 2.3 which provides a formula to compute the coefficients $c(w, w')$. This proof will be done in next section.

Let \mathcal{S}_w be a Schubert variety (of any isotropic or orthogonal grassmannian) and the associated double partition $\Lambda = (\alpha|\lambda), [\alpha|\lambda]$, where $\alpha = (\alpha_1 \geq \dots \geq \alpha_k > 0)$ and $\lambda = (\lambda_1 > \dots > \lambda_r > 0)$ which $\alpha_k \geq r$. Given a Schubert variety $\mathcal{S}_{w'}$ with partition $\Lambda' = (\alpha'|\lambda')$ such that $\ell(w) = \ell(w') + 1$, the partition w' is called

- (1) a α -removing of w if $\alpha' = (\alpha_1, \dots, \alpha_t - 1, \dots, \alpha_k)$ for some $1 \leq t \leq k$;
- (2) a λ -removing of w if $\lambda' = (\lambda_1, \dots, \lambda_t - 1, \dots, \lambda_r)$ for some $1 \leq t \leq r$.

When we look at the diagram, removing a box from a row of the partition Λ means that we are removing the last box of the respective row. In respect to the diagram, there are three different kinds of a λ -removing of w :

- (i) λ -removing of w from the diagonal (only happens for type C and B);
- (ii) λ -removing of w from a related column;

(iii) λ -removing of w from a non-related column;

Observe that for any λ -removing for groups of type C and B, by symmetry, we have $\lambda_t - 1$ unfilled boxes in the staircase pattern below the removed box. Being this a related column or not, u_p or v_q , is equal to $\lambda_t - 1$. The same idea holds for type D. This implies the following lemma:

Lemma 5.1. *Suppose that w' is a λ -removing of w and $\lambda'_t = \lambda_t - 1$ for some $1 \leq t \leq r$. Recall the formulas (6), for type C and B, and (12), for type D, of w in terms of the lengths u 's and v 's.*

- If it is a λ -removing from a related column, then

Type C or B: There is $p \in \{1, \dots, k\}$ such that $u_p = \lambda_t - 1$;

Type D: There is $p \in \{1, \dots, k\}$ such that $u_p = \lambda_t$.

- If it is λ -removing from a non-related column, then

Type C or B: There is $q \in \{1, \dots, n - k - r\}$ such that $v_q = \lambda_t - 1$;

Type D: There is $q \in \{1, \dots, n + 1 - k - r\}$ such that $v_q = \lambda_t$.

Theorem 5.2. *Given w, w' in $\mathcal{W}^{(k)}$ such that $\ell(w) = \ell(w') - 1$, this can be either a α -removing or a λ -removing of w and consider $t = t(w, w')$ the row where the box is removed. The coefficient $c(w, w')$ in the boundary map of the cellular homology of the respective G/P_Θ is given by:*

$$(18) \quad c(w, w') = (-1)^{\chi(w, w')} \cdot (1 + (-1)^{\kappa(w, w')})$$

where

$$\chi(w, w') = \begin{cases} 1 + \alpha_{t+1} + \dots + \alpha_k + |\lambda| & , \text{ if } w' \text{ is a } \alpha\text{-removing of } w; \\ 1 + \lambda_{t+1} + \dots + \lambda_r & , \text{ if } w' \text{ is a } \lambda\text{-removing of } w; \end{cases}$$

and $\kappa(w, w') = t + A(t)$ depends on the group G and which box is removed:

$$\textbf{Type C: } A(t) = \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ k & , \text{ for a } \lambda\text{-removing and } \lambda_t = 1; \\ 2k - p + 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = u_p \text{ for some } p; \\ k + n - q + 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = v_q \text{ for some } q. \end{cases}$$

$$\textbf{Type B: } A(t) = \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ 2k + t - 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t = 1; \\ 2k - p & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = u_p \text{ for some } p; \\ k + n - q & , \text{ for a } \lambda\text{-removing and } \lambda_t - 1 = v_q \text{ for some } q. \end{cases}$$

$$\textbf{Type D: } A(t) = \begin{cases} \alpha_t - 1 & , \text{ for a } \alpha\text{-removing;} \\ 2k - p - 1 & , \text{ for a } \lambda\text{-removing and } \lambda_t = u_p \text{ for some } p; \\ k + n - q & , \text{ for a } \lambda\text{-removing and } \lambda_t = v_q \text{ for some } q. \end{cases}$$

6. PROOF OF THEOREM 5.2

6.1. Type C and B. To get the boundary operator, we must compute $\phi(w) - \phi(w')$ where $\phi(w) = \sum_{\alpha \in \Pi_w} \alpha$ is the sum of the elements in the β -sequence of w .

Therefore, we have four different possibilities to remove a box from Λ as listed below. In these computations we only need to know what happens to the Top part since in [19] we can find the proof for the Bottom part. Remember that the permutation of w can be written as

$$w_{u, \lambda} = (u_k, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_r}, v_{n-k-r}, \dots, v_1)$$

and these values are obtained by counting the length of related and non-related columns of the bottom diagrams.

6.1.1. Case 1: α -removing of w . Suppose that we are removing a box from the row α_t for some $t \in \{1, \dots, k\}$. Looking through an imaginary diagonal line, the row α_t is related to the $(\alpha_t + k - t + 1)$ -th column in the bottom diagram D'_λ . Clearly, the $(\alpha_t + k - t)$ -th column must be non-related.

Call by u_p and v_q the length of related column $\alpha_t + k - t + 1$ and non-related column $\alpha_t + k - t$, respectively, where $p = t$ and $q = n + 1 - k - \alpha_t$.

After remove the box (t, α_t) , the related $(\alpha_t + k - t + 1)$ -th and non-related $(\alpha_t + k - t)$ -th columns of w swicht their role in the permutation and they become, respectively, non-related and related columns of w' . Then, the permutation of w' is obtained from w just by swapping u_p and v_q

$$w' = (u_k, \dots, v_q, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_r}, v_{n-k-r}, \dots, u_p, \dots, v_1).$$

Denote by β' the β -sequences associated to w' . Using Proposition 4.1, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those box at t -th row or α_t -th column. By definition,

$$\begin{aligned} w(k - t + 1) &= u_p; & w(k + \alpha_t) &= v_q; \\ w'(k - t + 1) &= v_q; & w'(k + \alpha_t) &= u_p. \end{aligned}$$

Then,

$$\begin{aligned} \beta_{i, \alpha_t}^T &= \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+\alpha_t)} = \varepsilon_{w(k-i+1)} - \varepsilon_{v_q}, \quad 1 \leq i \leq t; \\ \beta_{t, j}^T &= \varepsilon_{w(k-t+1)} - \varepsilon_{w(k+j)} = \varepsilon_{u_p} + \varepsilon_{w(k-j+1)}, \quad 1 \leq j \leq \alpha_t; \\ (\beta')_{i, \alpha_t}^T &= \varepsilon_{w(k-i+1)} - \varepsilon_{w(k+\alpha_t)} = \varepsilon_{w(k-i+1)} - \varepsilon_{u_p}, \quad 1 \leq i \leq t - 1; \\ (\beta')_{t, j}^T &= \varepsilon_{w(k-t+1)} - \varepsilon_{w(k+j)} = \varepsilon_{v_q} + \varepsilon_{w(k-j+1)}, \quad 1 \leq j \leq \alpha_t - 1. \end{aligned}$$

The contribution for $\phi(w) - \phi(w')$ occurs only at the t -th row and α_t -th column of the top diagram, so

$$\begin{aligned} \beta_{i, \alpha_t}^T - (\beta')_{i, \alpha_t}^T &= \varepsilon_{u_p} - \varepsilon_{v_q}, \quad \text{for all } 1 \leq i \leq t - 1; \\ \beta_{t, j}^T - (\beta')_{t, j}^T &= \varepsilon_{u_p} - \varepsilon_{v_q}, \quad \text{for all } 1 \leq j \leq \alpha_t - 1; \\ \beta_{t, \alpha_t}^T &= \varepsilon_{u_p} - \varepsilon_{v_q}, \quad \text{for } (t, \alpha_t). \end{aligned}$$

From now on, it is useful to write $\phi(w) - \phi(w') = S^T(w, w') + S^B(w, w')$, where

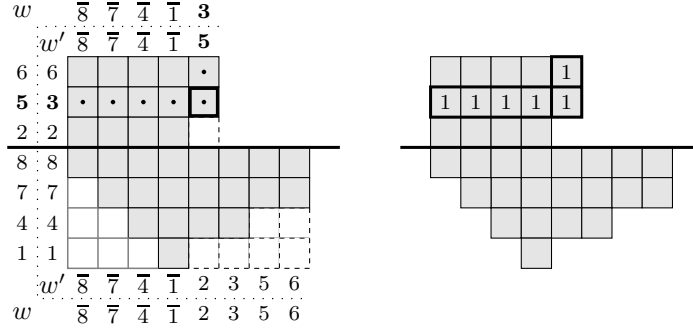
$$(19) \quad S^T(w, w') = \sum_{(i, j) \in D_\alpha} \beta_{i, j}^T - \sum_{(i, j) \in D_{\alpha'}} (\beta')_{i, j}^T,$$

$$(20) \quad S^B(w, w') = \sum_{(i, j) \in D_\lambda} \beta_{i, j}^B - \sum_{(i, j) \in D_{\lambda'}} (\beta')_{i, j}^B.$$

Then, $S^T(w, w') = (t - 1)(\varepsilon_{u_p} - \varepsilon_{v_q}) + (\alpha_t - 1)(\varepsilon_{u_p} - \varepsilon_{v_q}) + (\varepsilon_{u_p} - \varepsilon_{v_q}) = (t + \alpha_t - 1)(\varepsilon_{u_p} - \varepsilon_{v_q})$ and $S^B(w, w') = 0$, since $w_\lambda = w'_\lambda$.

Therefore, $\phi(w) - \phi(w') = (t + \alpha_t - 1)(\varepsilon_{u_t} - \varepsilon_{v_{n+1-k-\alpha_t}})$. This formula for the coefficient can be computed just looking to the diagram of w and it will be explored in the next example.

Example 6.1. Consider $\Lambda = (5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$ and its β -sequences in Figure 8. Suppose that $\Lambda' = (5, 4, 4|8, 7, 4, 1)$, where $t = 2$. The β -sequences for w' can be described in the diagram as same as the ones for w . Comparing


 FIGURE 11. $\kappa(w, w')$ for $\Lambda = (5, 5, 4|8, 7, 4, 1)$ and $\Lambda' = (5, 4, 4|8, 7, 4, 1)$.

both diagrams, we can see in Figure 11 (left) that the β -sequences filled in the boxes are only different for some in the top part which have a dot. Subtracting the diagram of β -sequence of w' from the diagram of w , we get an integer multiple of the root $\varepsilon_{u_p} - \varepsilon_{v_q} = \varepsilon_5 - \varepsilon_3$ in the dotted boxes and zero in all other. In the Figure 11 (right), the multiplicity of such root is 1 for the removed box and for all in the same row or same column in the top part. Then, $\phi(w) - \phi(w')$ can be easily computed just using the diagram by adding all coefficients in it. For this example, $\phi(w) - \phi(w') = (t + \alpha_t - 1)(\varepsilon_5 - \varepsilon_3) = 6(\varepsilon_5 - \varepsilon_3)$.

The above process can be applied in general for any α -removing of w : put 1's into all boxes of the top part in the same column and row of the removed box. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = t + \alpha_t - 1$.

Note that the root $\varepsilon_{u_t} - \varepsilon_{v_{n+1-k-\alpha_t}}$ can be a non-simple positive root. No other case for groups of type C and B has this particularity.

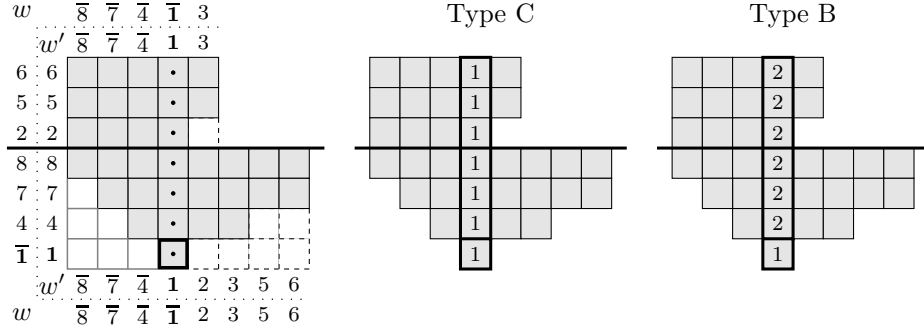
Remark 6.2. Although this whole work is devoted to establish the relation between the half-shifted Young diagrams and Grassmannians for groups of type B, C and D, we can point out that the Schubert varieties of the usual Grassmannian (of type A) are parametrized by the standard Young diagram, a particular case of half-shifted when the bottom part $\lambda = \emptyset$. Then, the homology coefficients for the usual Grassmannians can be computed considering only the α -removing case as made above.

6.1.2. *Case 2: λ -removing in a diagonal of w .* Suppose that we are removing a box from the row λ_t for some $t \in \{1, \dots, r\}$ such that this is in the diagonal of the diagram D'_λ of w . So, we must have $t = r$ and $\lambda_r = 1$. Since the new “free” of w' box is in the first non-related column, the permutations w and w' are

$$\begin{aligned} w &= (u_k, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_{r-1}}, \overline{1}, v_{n-k-r}, \dots, v_1), \\ w' &= (u_k, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_{r-1}}, 1, v_{n-k-r}, \dots, v_1). \end{aligned}$$

By Proposition 4.1, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those in the r -th column. Observe that, since $\alpha_k \geq r$ then all the boxes in the column r must belong to the diagram D_α . Then,

$$\begin{aligned} \beta_{i,r}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_1, \quad 1 \leq i \leq k; \\ (\beta')_{i,r}^T &= \varepsilon_{w(k-i+1)} - \varepsilon_1, \quad 1 \leq i \leq k. \end{aligned}$$

FIGURE 12. $\kappa(w, w')$ for $\Lambda = (5, 5, 4|8, 7, 4, 1)$ and $\Lambda' = (5, 5, 4|8, 7, 4)$.

For the top part, $S^T(w, w') = 2k\varepsilon_1$. Now, we must consider the type C and B separately:

- **Type C:** For the bottom part, $S^B(w, w) = r(2\varepsilon_1) = t(2\varepsilon_1)$. Therefore, $\phi(w) - \phi(w') = (t + k)(2\varepsilon_1)$.
- **Type B:** For the bottom part, $S^B(w, w) = (2r - 1)\varepsilon_1 = (2t - 1)\varepsilon_1$. Therefore, $\phi(w) - \phi(w') = (2k + 2t - 1)\varepsilon_1$.

Example 6.3. Consider $\Lambda = (5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$ (Type C) and suppose that $\Lambda' = (5, 5, 4|8, 7, 4)$, where $t = 4$. Subtracting the diagram of β -sequence of w' from w , we get an integer multiple of the root $2\varepsilon_1$ into the dotted boxes, which is given in the Figure 12 (center). Then, the sum of such multiplicities is $\phi(w) - \phi(w') = (t + k)(2\varepsilon_1) = 7(2\varepsilon_1)$. If Λ is in $\text{OG}(5, 15)$ (Type B), the integer multiple of the root ε_1 is given in the Figure 12 (right) and $\phi(w) - \phi(w') = (2k + 2t - 1)\varepsilon_1 = 13\varepsilon_1$.

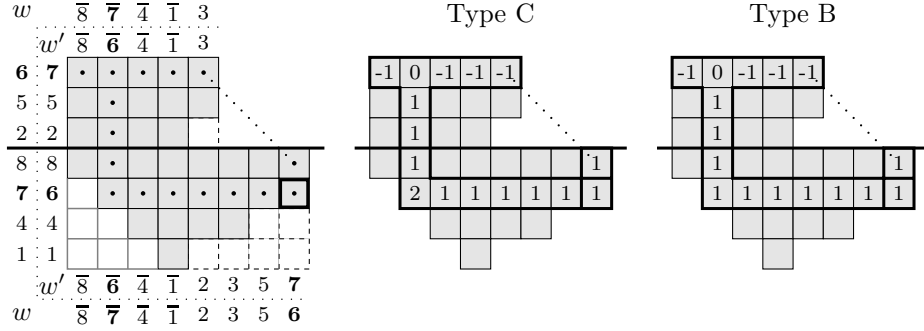
The above process can be applied in general for any λ -removing of a diagonal column of w : for groups of type C, put 1's into all boxes in the same column of the removed box, in both top and bottom parts. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = t + k$. In the other hand, for groups of type B, put 1 only into the removed box and 2's into all boxes above and in the same column of the removed box. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = 2k + 2t - 1$.

6.1.3. Case 3: λ -removing of a related column of w . Suppose that we are removing a box from the row λ_t for some $t \in \{1, \dots, r\}$ such that this belongs to a related column of the diagram of w . Using Lemma 5.1, there exist a $p \in \{1, \dots, k\}$ such that $u_p = \lambda_t - 1$. When the box (t, λ_t) is removed from the bottom diagram of w , it will increase by 1 the length u_p and the permutation w' is

$$w' = (u_k, \dots, u_p + 1, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_t - 1}, \dots, \overline{\lambda_r}, v_{n-k-r}, \dots, v_1).$$

Using Proposition 4.1, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those box in p -th row or t -th column. Then,

$$\begin{aligned} \beta_{i,t}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t}, \quad 1 \leq i \leq k; & (\beta')_{i,t}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t-1}, \quad 1 \leq i \leq k; \\ \beta_{p,j}^T &= \varepsilon_{\lambda_t-1} - \varepsilon_{w(k+j)}, \quad 1 \leq j \leq \alpha_p; & (\beta')_{p,j}^T &= \varepsilon_{\lambda_t} - \varepsilon_{w(k+j)}, \quad 1 \leq j \leq \alpha_p. \end{aligned}$$


 FIGURE 13. $\kappa(w, w')$ for $\Lambda = (5, 5, 4|8, 7, 4, 1)$ and $\Lambda' = (5, 5, 4|8, 6, 4, 1)$.

Recall that, by definition, $\alpha_p = \lambda_t + p - k - 2 + d_p$, where $d_p = \#\{j \mid \lambda_j > u_p\} = t$. Then, for the top part, $S^T(w, w') = (k - \alpha_p)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}}) = (2k - \lambda_t - p - t + 2)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. Now, we must consider the type C and B separately:

- **Type C:** For the bottom part, $S^B(w, w) = (\lambda_t + 2t - 1)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. Then, $\phi(w) - \phi(w') = (2k - p + t + 1)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$.
- **Type B:** For the bottom part, $S^B(w, w) = (\lambda_t + 2t - 2)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. Then, $\phi(w) - \phi(w') = (2k - p + t)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$.

Example 6.4. Consider $\Lambda = (5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$ and suppose that $\Lambda' = (5, 5, 4|8, 6, 4, 1)$, where $t = 2$ and $p = 1$. Subtracting the diagram of β -sequence of w' from w , we get an integer multiple of the root $\varepsilon_7 - \varepsilon_6$ into the dotted boxes, which is given in the Figure 13 (center). Then, the sum of such multiplicities is $\phi(w) - \phi(w') = (2k - p + t + 1)(\varepsilon_7 - \varepsilon_6) = 8(\varepsilon_7 - \varepsilon_6)$. If Λ is in $\text{OG}(5, 15)$ (Type B), the integer multiple of the root $\varepsilon_7 - \varepsilon_6$ is given in the Figure 13 (right) and $\phi(w) - \phi(w') = (2k - p + t)(\varepsilon_7 - \varepsilon_6) = 7(\varepsilon_7 - \varepsilon_6)$.

The above process can be applied in general for any λ -removing of a related column of w : for type C, fill the diagram in the same way; for type B, just change the 2 in the diagram by 1. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = 2k - p + t + 1$ for type C and $\kappa(w, w') = 2k - p + t$ for type B.

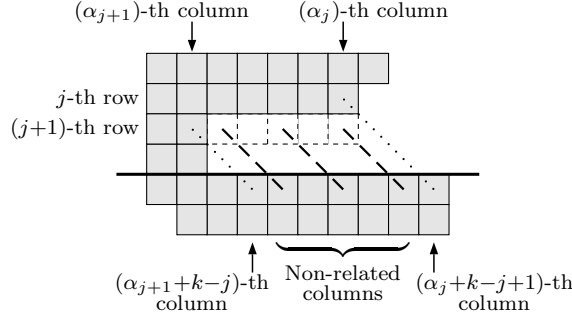
6.1.4. Case 4: λ -removing of a non-related column. Suppose that we are removing a box from the row λ_t for some $t \in \{1, \dots, r\}$ such that this belongs to a non-related column of the diagram of w .

Using Lemma 5.1, there exist a $q \in \{1, \dots, n - r - k\}$ such that $v_q = \lambda_t - 1$. Removing the box (t, λ_t) from the bottom diagram of w , it will increase by 1 the length of v_q and the permutation w' is

$$w' = (u_k, \dots, u_1, \overline{\lambda_1}, \dots, \overline{\lambda_t - 1}, \dots, \overline{\lambda_r}, v_{n-k-r}, \dots, v_q + 1, \dots, v_1)$$

where $v_q + 1$ is in the position $n - q + 1$ in the permutation.

Using Proposition 4.1, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those box in t -th or $(n - k - q + 1)$ -th columns.

FIGURE 14. Relation between columns of α and columns of λ .

Define the transpose partition $\alpha^T \in \mathcal{R}(n-k, k)$ of α by $\alpha_i^T = \#\{j \mid \alpha_j \geq i\}$, for all $1 \leq i \leq n-k$. To simplify, call $m = n-k-q+1$. Then,

$$\begin{aligned} \beta_{i,t}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_t}, \quad 1 \leq i \leq k; & (\beta')_{i,t}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_{\lambda_{t-1}}, \quad 1 \leq i \leq k; \\ \beta_{i,m}^T &= \varepsilon_{w(k-i+1)} - \varepsilon_{\lambda_{t-1}}, \quad 1 \leq i \leq \alpha_m^T; & (\beta')_{i,m}^T &= \varepsilon_{w(k-i+1)} - \varepsilon_{\lambda_t}, \quad 1 \leq j \leq \alpha_m^T. \end{aligned}$$

For the top part, $S^T(w, w') = (k + \alpha_m^T)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. We can rewrite the value of $S^T(w, w')$ using the following proposition:

Proposition 6.5. *Let $\Lambda = (\alpha \mid \lambda)$ be a partition associated to w . The transpose permutation α^T belonging to $\mathcal{R}(n-k, k)$ can be written as*

$$\alpha_i^T = \begin{cases} k & , \text{ if } 1 \leq i \leq r; \\ -v_{n-k-i+1} + i + k - \tilde{d}_i & , \text{ if } r < i \leq n-k; \end{cases}$$

where $\tilde{d}_i = \#\{l \mid \lambda_l > v_{n-k-i+1}\}$.

Proof. By definition of Λ , we know that $\alpha_k \geq r$. If $1 \leq i \leq r$ then clearly $\alpha_i^T = k$, the number of rows of α .

Now, if $r < i \leq n-k$, we must analyze the diagram differently. First of all, two any consecutive rows j and $j+1$ are related, respectively, to the $(\alpha_j + k - j + 1)$ -th and $(\alpha_{j+1} + k - j)$ -th columns in the bottom part. There are exactly $\alpha_j - \alpha_{j+1}$ non-related columns between them.

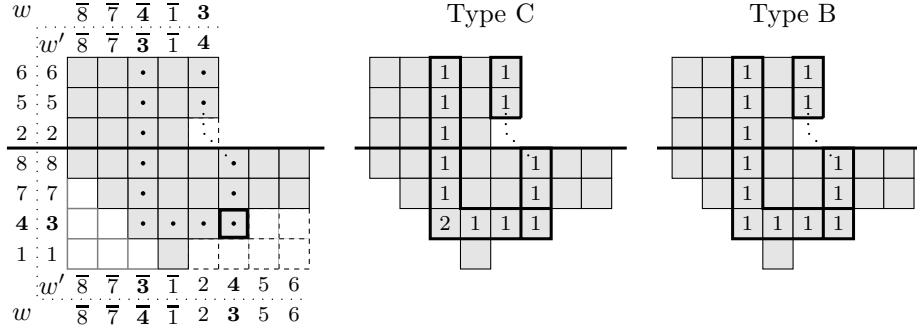
So, we can relate each box $(j+1, t)$, for $t \in \{\alpha_{j+1} + 1, \dots, \alpha_j\}$, to a non-related column between $(\alpha_j + k - j + 1)$ -th and $(\alpha_{j+1} + k - j)$ -th columns by drawing a diagonal line from the box $(j+1, t)$ to a first box in the $(t + k - j)$ -th column. The Figure 14 shows this relation to non-related columns using dashed lines.

In the other hand, we can observe that each column t of the top part, for $r < t \leq n-k$, is related, by the above process, to some non-related column in the bottom part, and such column is exactly the $(v_{n-k-t+1} + \tilde{d}_t)$ -th column of λ .

Hence, for the column i of α , we have the equality $i + k - \alpha_i^T = v_{n-k-i+1} + \tilde{d}_i$. \square

Since $m \geq r + 1$, then $\alpha_m^T = -\lambda_t + n - q - t + 2$, implying that $S^T(w, w') = (k - \lambda_t + n - q - t + 2)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. Now, we must consider the type C and B separately:

- **Type C:** For the bottom part, $S^B(w, w) = (\lambda_t + 2t - 1)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$. Then, $\phi(w) - \phi(w') = (k + n - q + t + 1)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_{t-1}})$.


 FIGURE 15. $\kappa(w, w')$ for $\Lambda = (5, 5, 4|8, 7, 4, 1)$ and $\Lambda' = (5, 5, 4|8, 7, 3, 1)$.

- **Type B:** For the bottom part, $S^B(w, w) = (\lambda_t + 2t - 2)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t - 1})$. Then, $\phi(w) - \phi(w') = (k + n - q + t)(\varepsilon_{\lambda_t} - \varepsilon_{\lambda_t - 1})$.

Example 6.6. Consider $\Lambda = (5, 5, 4|8, 7, 4, 1)$ in $\text{IG}(5, 16)$ and suppose that $\Lambda' = (5, 5, 4|8, 7, 3, 1)$, where $t = 3$ and $q = 1$. Subtracting the diagram of β -sequence of w' from w , we get an integer multiple of the root $\varepsilon_4 - \varepsilon_3$ into the dotted boxes, which is given in the Figure 15 (center). Then, the sum of such multiplicities is $\phi(w) - \phi(w') = (k + n - q + t + 1)(\varepsilon_4 - \varepsilon_3) = 14(\varepsilon_4 - \varepsilon_3)$. If Λ is in $\text{OG}(5, 15)$ (Type B), the integer multiple of the root $\varepsilon_4 - \varepsilon_3$ is given in the Figure 15 (right) and $\phi(w) - \phi(w') = (k + n - q + t)(\varepsilon_4 - \varepsilon_3) = 13(\varepsilon_4 - \varepsilon_3)$.

The above process can be applied in general for any λ -removing of a non-related column of w : for type C, fill the diagram in the same way; for type B, just change the 2 in the diagram by 1. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = k + n - q + t + 1$ for type C and $\kappa(w, w') = k + n - q + t$ for type B.

6.2. Type D. The boundary operator is obtained once we compute $\phi(w) - \phi(w')$ where $\phi(w) = \sum_{\alpha \in \Pi_w} \alpha$ is the sum of the elements in the β -sequence of w . The approach for the type D is almost the same as for the types B, C. In some cases, the calculations are exactly the same. However, the existence of elements of type 0, 1 and 2 asks for a careful analysis in some specific situations which turn out to be our focus in this section. We use the results obtained by [19] for the bottom part when needed and compute here the top part.

Remember that the permutation of w can be written as

$$w_{u, \lambda} = (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_1)$$

and these values are obtained by counting the length of related and non-related columns of the bottom part of the diagram.

6.2.1. Case 1: α -removing of w . Suppose that we are removing a box from the row α_t for some $t \in \{1, \dots, k\}$. We first observe that the same pattern over the diagrams verified in the types B, C occur here, namely, this removing transforms a related column and a non-related column, respectively, into a non-related and related column. Explicitly, if u_p , $1 \leq p \leq k$, and v_q , $1 \leq q \leq n + 1 - k - r$, are the length of related column $\alpha_t + k - t + 1$ and non-related column $\alpha_t + k - t$,

respectively, we have that

$$\begin{aligned} w &= (\hat{u}_k, \dots, u_p, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_q, \dots, v_1), \\ w' &= (\hat{u}_k, \dots, v_q, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, u_p, \dots, v_1). \end{aligned}$$

Observe also that $p = t$ and $q = n + 2 - k - \alpha_t$. In terms of the beta-sequences associated to the diagrams, the Proposition 4.3 shows that the formula for the roots in the Top diagram is the same as for the type B, C given by the Proposition 4.1.

However, notice that in the context of type D changes in the last row may reflect in a change over the type of the element. This implies that we must divide this analysis into two cases:

Case 1a: $t < k$ and if $t = k$, $\alpha_k > r + 1$, where $r = \ell(\lambda)$.

These cases are characterized by the fact that the type of w remains unchanged after removing the box.

- (1) If $\text{type}(w) = 0$ then we only have that $t < k$, otherwise, if $t = k$ and $\text{type}(w) = 0$ then $\alpha_k = r = \ell(\lambda)$ is minimal such that it is not possible to remove a box in the last row. Since $p = t$, we have that u_k is unchanged which means that the type is preserved.
- (2) If $\text{type}(w) \neq 0$ then we must have $\alpha_k > r + 1$ (both $t < k$ and $t = k$ cases are included). But since $q = n + 2 - k - \alpha_t$, $\alpha_k > r + 1$ implies that $1 \leq q < n + 1 - k - r$, i.e., the term $v_{n+1-k-r}$ which characterizes the type of w as 1 or 2 is preserved.

As a consequence, by the considerations above, we get exactly the same results as those for the type B, C case. Hence, $\phi(w) - \phi(w') = (t + \alpha_t - 1)(\varepsilon_{u_t} - \varepsilon_{v_{n+2-k-\alpha_t}})$.

Case 1b: $t = k$ and $\alpha_k = r + 1$, where $r = \ell(\lambda)$.

These cases are characterized by the fact that elements of type 1 and 2 are changed to a type 0 element after removing the box. Indeed, if $(\alpha, \lambda) = (\alpha_1, \dots, \alpha_k, \lambda_1, \dots, \lambda_r)$ is the partition for $w_{(\alpha, \lambda)}$, with $\alpha_k = r + 1$ and $r = \ell(\lambda)$, then $w'_{(\alpha', \lambda)}$ will be the permutation given by the partition $(\alpha', \lambda) = (\alpha_1, \dots, \alpha_k - 1, \lambda_1, \dots, \lambda_r)$, i.e., w is an element of type 1, 2 and w' is an element of type 0.

In order to facilitate the calculations for this case and the next ones, define $\hat{1}$ and $\check{1}$ as follows

$$(21) \quad \hat{1} := \begin{cases} 1 & , \text{ if } r \text{ is even,} \\ \bar{1} & , \text{ if } r \text{ is odd.} \end{cases} ; \quad \check{1} := \begin{cases} 1 & , \text{ if } \text{type}(w) = 1, \\ \bar{1} & , \text{ if } \text{type}(w) = 2. \end{cases}$$

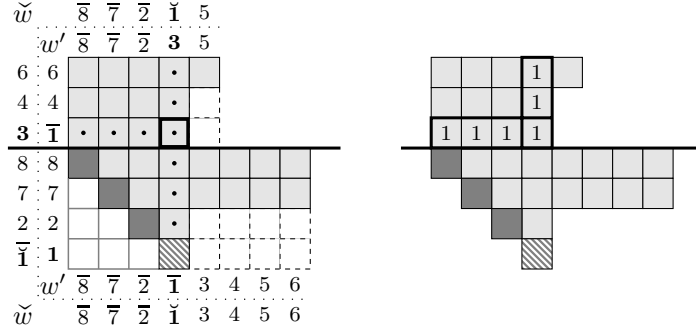
Then, the permutations w and w' are

$$\begin{aligned} w &= (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, \check{1}, \dots, v_1), \\ w' &= (\hat{1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, u_k, \dots, v_1). \end{aligned}$$

The beta sequences of β and β' of w and w' , respectively, are represented in the corresponding diagram according to the Proposition 4.3. We describe below only the boxes that are changed and contribute to the homology coefficient, namely, the k -row and the $(r + 1)$ -column of the top diagram. At these rows and columns we have the following roots:

$$\begin{aligned} \beta_{i, r+1}^T &= \varepsilon_{u_i} - \varepsilon_{\check{1}}, \quad 1 \leq i \leq k; & \beta'_{i, r+1}^T &= \varepsilon_{u_i} - \varepsilon_{u_k}, \quad 1 \leq i \leq k - 1; \\ \beta_{k, j}^T &= \varepsilon_{u_k} - \varepsilon_{\lambda_j}, \quad 1 \leq j \leq \alpha_k = r + 1; & \beta'_{k, j}^T &= \varepsilon_{\hat{1}} + \varepsilon_{\lambda_j}, \quad 1 \leq j \leq \alpha_k - 1 = r. \end{aligned}$$

implying that $S^T(w, w') = k(\varepsilon_{u_k} - \varepsilon_{\check{1}}) + r(\varepsilon_{u_k} - \varepsilon_{\hat{1}})$.


 FIGURE 16. $\kappa(w, w')$ for $\Lambda = [5, 4, 4|7, 6, 1]$ and $\Lambda' = (5, 4, 3|7, 6, 1)$.

The elements w_λ and w'_λ are given by

$$w_\lambda = (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, \check{1}, u_1, \dots, u_k, \dots, v_1),$$

$$w'_\lambda = (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_r + 1}, \hat{1}, u_1, \dots, u_k, \dots, v_1)$$

and, by Proposition 4.3, at the $(r+1)$ -column of the bottom diagram, for $1 \leq i \leq r$,

$$\beta_{i,r+1}^B = \varepsilon_{\check{w}_\lambda(2n+3-i)} - \varepsilon_{\check{w}_\lambda(r+1)} = \varepsilon_{v_i} - \varepsilon_{\check{1}},$$

$$\beta_{i,r+1}'^B = \varepsilon_{w'_\lambda(2n+3-i)} - \varepsilon_{w'_\lambda(r+1)} = \varepsilon_{v_i} - \varepsilon_{\hat{1}}$$

implying that $S^B(w, w') = r(\varepsilon_{\hat{1}} - \varepsilon_{\check{1}})$.

For sake of illustration, let us consider the case when $\text{type}(w) = 1$ and $\ell(\lambda)$ is odd, i.e., $\check{1} = 1$ and $\hat{1} = -1$. There are r roots $\varepsilon_{u_k} + \varepsilon_1$ given by the difference of roots at the r -th row of the top diagram. There are also r roots $-2\varepsilon_1$ given by the difference of the roots at the r -column of the bottom diagram. Adding them up we have r roots

$$(\varepsilon_{u_k} + \varepsilon_1) - 2\varepsilon_1 = \varepsilon_{u_k} - \varepsilon_1$$

which correspond to the same roots given by the difference of roots at the $(r+1)$ -th column of the top diagram. Analogous considerations will hold when $\text{type}(w) = 2$ and $\ell(\lambda)$ is even.

Therefore, we will have $\phi(w) - \phi(w') = (k+r)(\varepsilon_{u_k} - \varepsilon_{\check{1}}) = (t + \alpha_t - 1)(\varepsilon_{u_k} - \varepsilon_{\check{1}})$.

To illustrate this case, we will consider an example where the type of the permutation changes when we remove some box.

Example 6.7. First of all, consider $\Lambda_1 = [5, 4, 4|7, 6, 1]$ in $\text{OG}(5, 16)$ of type 1 and suppose that $\Lambda' = (5, 4, \mathbf{3}|7, 6, 1)$ of type 0, where $t = 3$. Subtracting the diagram of β -sequence of w'_1 from w , we get: (i) for the top part, the sum of the roots is $S^T(w_1, w') = 3(\varepsilon_3 - \varepsilon_1) + 3(\varepsilon_3 + \varepsilon_1) = 6\varepsilon_3$ and (ii) for the bottom part, the sum of the roots is $S^B(w_1, w') = 6\varepsilon_1$. This implies that $\phi(w_1) - \phi(w') = (t + \alpha_t - 1)(\varepsilon_3 - \varepsilon_1) = 6(\varepsilon_3 - \varepsilon_1)$. In the other hand, considering $\Lambda_2 = (5, 4, 4|7, 6, 1]$ of type 2 and the same Λ' , then (i) for the top part, the sum of the roots is $S^T(w_2, w') = 6(\varepsilon_3 - \varepsilon_1)$ and (ii) for the bottom part, the sum of the roots is $S^B(w_2, w') = 0$, implying that $\phi(w_2) - \phi(w') = (t + \alpha_t - 1)(\varepsilon_3 - \varepsilon_1) = 6(\varepsilon_3 - \varepsilon_1)$.

Even though the above process does not give exactly the root $\varepsilon_{u_k} - \varepsilon_{\check{1}}$ as subtraction of β -sequences, the coefficient κ is given in the same way as the α -removing case for groups of type C and B: put 1's into all boxes of the top part in the same

column or row of the removed box. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = t + \alpha_t - 1$.

6.2.2. Case 2: λ -removing of a related column. Suppose that we are removing a box from the row λ_t for some $t \in \{1, \dots, r\}$ such that this belongs to a related column of w . By the Lemma 5.1, there is a $p \in \{1, \dots, k\}$ such that $u_p = \lambda_t$. Lets considering two cases: when $\lambda_t = 1$ and when it is not.

Case 2a: $\lambda_t = 1$.

This case means we are removing the unique box at the last row of λ and w must type 0. Besides, $t = r$ and $p = k$.

After remove the unique box at the row λ_r , we have $\ell(\lambda') < a_k$, which implies that w' does not have type 0. Moreover, the simple permutation of the map s^B at the removed box of w depends on the parity of $r = \ell(\lambda)$: it is s_1 if r is even and s_0 if r is odd. Then, the simple permutation at the first box of the row $r - 1 = \ell(\lambda')$ is s_0 if r is even and s_1 if r is odd, implying that w' has type 2 and 1, respectively, by Remark 3.3.

We can write the permutation of w and w' as

$$\begin{aligned} w &= (\hat{1}, u_{k-1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{2}, v_{n+1-k-r}, \dots, v_1), \\ w' &= (\bar{2}, u_{k-1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{1}, v_{n+1-k-r}, \dots, v_1) \end{aligned}$$

since this new “free” box is in the first related column and has length 2. By Proposition 4.3, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those in the last row and in the r -th column. Therefore,

$$\begin{aligned} \beta_{i,r}^T &= \varepsilon_{w(k-i+1)} + \varepsilon_2, \quad 1 \leq i \leq k; & (\beta')_{i,r}^T &= \varepsilon_{\tilde{w}'(k-i+1)} + \varepsilon_{\hat{1}}, \quad 1 \leq i \leq k; \\ \beta_{k,j}^T &= \varepsilon_{\hat{1}} - \varepsilon_{w(k+j)}, \quad 1 \leq j \leq \alpha_k = r; & (\beta')_{k,j}^T &= \varepsilon_2 - \varepsilon_{\tilde{w}'(k+j)}, \quad 1 \leq j \leq \alpha_k = r. \end{aligned}$$

Then, $S^T(w, w') = (k-r)(\varepsilon_2 - \varepsilon_{\hat{1}})$. Now, the β -sequence for the bottom diagram change only for the r -th and $(r+1)$ -th columns. In fact, the permutations are

$$\begin{aligned} w_\lambda &= (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{2}, \hat{1}, \dots), \\ \tilde{w}'_\lambda &= (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{1}, 2, \dots) \end{aligned}$$

and the β sequences are:

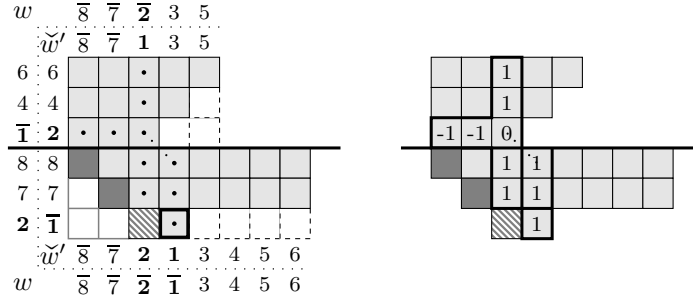
$$\begin{aligned} \beta_{i,r}^B &= \varepsilon_{w_\lambda(2n+3-i)} + \varepsilon_2, \quad 1 \leq i \leq r-1; & (\beta')_{i,r}^B &= \varepsilon_{\tilde{w}'_\lambda(2n+3-i)} + \varepsilon_{\hat{1}}, \quad 1 \leq i \leq r-1; \\ \beta_{i,r+1}^B &= \varepsilon_{w_\lambda(2n+3-i)} - \varepsilon_{\hat{1}}, \quad 1 \leq i \leq r; & (\beta')_{i,r+1}^B &= \varepsilon_{\tilde{w}'_\lambda(2n+3-i)} - \varepsilon_2, \quad 1 \leq i \leq r. \end{aligned}$$

Then, $S^B(w, w') = (2r-1)(\varepsilon_2 - \varepsilon_{\hat{1}})$. Therefore, $\phi(w) - \phi(w') = (k+r-1)(\varepsilon_2 - \varepsilon_{\hat{1}}) = (2k-p+t-1)(\varepsilon_2 - \varepsilon_{\hat{1}})$.

Case 2b: $\lambda_t > 1$.

If $\text{type}(w) = 0$ then $p \neq k$ and w' still has type 0 because remove a box from $\lambda_t > 1$ cannot change the number of rows of λ' . Since $\lambda_t = u_p$, the permutations w and w' are

$$\begin{aligned} w &= (\hat{1}, \dots, \lambda_t, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_1), \\ w' &= (\hat{1}, \dots, \lambda_t + 1, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, v_1). \end{aligned}$$


 FIGURE 17. $\kappa(w, w')$ for $\Lambda = (5, 4, 3|7, 6, 1)$ and $\Lambda' = [5, 4, 3|7, 6)$.

Observe that the hat at the first element won't be a problem to compute the β -sequences using the Proposition 4.3 because $p \neq k$. Now, if $\text{type}(w) = 1$ or 2 then w' still has the same type of w and the permutations are

$$w = (\hat{u}_k, \dots, \lambda_t, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t + 1}, \dots, \overline{\lambda_r + 1}, \check{1}, v_{n-k-r}, \dots, v_1),$$

$$w' = (\hat{u}_k, \dots, \lambda_t + 1, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t}, \dots, \overline{\lambda_r + 1}, \check{1}, v_{n-k-r}, \dots, v_1).$$

Again, the hat at the first position won't affect the β -sequences because the Proposition 4.3 uses, for type 1 and 2, the permutations \check{w} and \check{w}' , which are, by definition, the same permutations but without the hat at the first position.

To compute the β -sequences of the Top and Bottom parts, we can use the same ideas as in Case 3 of type C and the results in [19] and recalling that $\alpha_p = u_p + p - k - 1 + \#\{j \mid \lambda_j + 1 > u_p\}$.

$$\text{Therefore, } \phi(w) - \phi(w') = (2k - p + t - 1)(\varepsilon_{\lambda_t+1} - \varepsilon_{\lambda_t}).$$

Example 6.8. Consider $\Lambda = (5, 4, 3|7, 6, 1)$ in $\text{OG}(5, 16)$ of type 0 and suppose that we are removing the element $\lambda_3 = 1$. Since the permutation of w can be written, by the row-reading, as $w = s_0 s_6 s_5 s_4 s_3 s_2 s_1 \dots$ then $w' = s_6 s_5 s_4 s_3 s_2 s_1 \dots$ and, by Remark 3.3, w' has type 1 and its partition is $\Lambda' = [5, 4, 3|7, 6)$, where $t = 3$ and $p = 3$. Then, we get: (i) for the top part, the sum of the roots is $S^T(w, w') = 0$ and (ii) for the bottom part, the sum of the roots is $S^B(w, w') = 5(\varepsilon_2 + \varepsilon_1)$. This implies that $\phi(w) - \phi(w') = (2k - p + t - 1)(\varepsilon_2 + \varepsilon_1) = 5(\varepsilon_2 + \varepsilon_1)$.

The above process gives exactly the root $\varepsilon_2 - \varepsilon_1$ as a subtraction of β -sequences, the coefficient κ is given in the same way as the λ -removing case of a related column for groups of type C and B: fill the diagram in the same way of above example. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = 2k - p + t - 1$.

6.2.3. Case 3: λ -removing of a non-related column. Suppose that we are removing a box from the row λ_t for some $t \in \{1, \dots, r\}$ such that this belongs to a non-related column of w . By the Lemma 5.1, there is a $q \in \{1, \dots, n + 1 - k - r\}$ such that $v_q = \lambda_t$. Lets considering two cases: when $\lambda_t = 1$ and when it is not.

Case 3a: $\lambda_t = 1$.

This case means we are removing the unique box at the last row of λ and w must have type 1 or 2 otherwise if w has type 0, the box lies in related column. Clearly, $t = r$ and $q = n + 1 - k - r$, as $v_q = \lambda_t = 1$.

By the row-reading map, if $\text{type}(w) = 1$ then $\text{type}(w') = 2$. In fact, $\text{type}(w) = 1$ implies that the removed box is labeled by s_1 and hence the diagonal of the last

row of w' is labeled by s_0 , which means that $\text{type}(w') = 2$. For the same reason, if $\text{type}(w) = 2$ then $\text{type}(w') = 1$. The permutation w is

$$w = (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{2}, \check{1}, v_{n-k-r}, \dots, v_1)$$

and when we remove the box $(r, r+1)$, the permutation of w' becomes

$$w' = (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{1}, 2, v_{n-k-r}, \dots, v_1).$$

By Proposition 4.3, we can observe that the β -sequences in the top diagram are equal to w and w' unless for those in the r -th and $(r+1)$ -th columns and both columns must have k box each because $\alpha_k > \ell(\lambda)$. Then

$$\begin{aligned} \beta_{i,r}^T &= \varepsilon_{\check{w}(k-i+1)} + \varepsilon_2, \quad 1 \leq i \leq k; & (\beta')_{i,r}^T &= \varepsilon_{\check{w}'(k-i+1)} + \varepsilon_{\check{1}}, \quad 1 \leq i \leq k; \\ \beta_{i,r+1}^T &= \varepsilon_{\check{w}(k-i+1)} - \varepsilon_{\check{1}}, \quad 1 \leq i \leq k; & (\beta')_{i,r+1}^T &= \varepsilon_{\check{w}'(k-i+1)} - \varepsilon_2, \quad 1 \leq i \leq k. \end{aligned}$$

Then, $S^T(w, w') = 2k(\varepsilon_2 - \varepsilon_{\check{1}})$. Now, the β -sequence for the bottom diagram change only for the r -th and $(r+1)$ -th columns. In fact, the permutations are

$$\begin{aligned} \check{w}_\lambda &= (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{2}, \check{1}, \dots), \\ \check{w}'_\lambda &= (\overline{\lambda_1 + 1}, \dots, \overline{\lambda_{r-1} + 1}, \bar{1}, 2, \dots) \end{aligned}$$

and the β sequences are:

$$\begin{aligned} \beta_{i,r}^B &= \varepsilon_{\check{w}_\lambda(2n+3-i)} + \varepsilon_2, \quad 1 \leq i \leq r-1; & (\beta')_{i,r}^B &= \varepsilon_{\check{w}'_\lambda(2n+3-i)} + \varepsilon_{\check{1}}, \quad 1 \leq i \leq r-1; \\ \beta_{i,r+1}^B &= \varepsilon_{\check{w}_\lambda(2n+3-i)} - \varepsilon_{\check{1}}, \quad 1 \leq i \leq r; & (\beta')_{i,r+1}^B &= \varepsilon_{\check{w}'_\lambda(2n+3-i)} - \varepsilon_2, \quad 1 \leq i \leq r. \end{aligned}$$

Then, $S^B(w, w') = (2r-1)(\varepsilon_2 - \varepsilon_{\check{1}})$. Hence, $\phi(w) - \phi(w') = (2k+2r-1)(\varepsilon_2 - \varepsilon_{\check{1}}) = (k+n-q+t)(\varepsilon_2 - \varepsilon_{\check{1}})$.

Case 3b: $\lambda_t > 1$.

If $\text{type}(w) = 0$ then w' still has type 0 because remove a box from $\lambda_t > 1$ cannot change the number of rows of λ' . Since $\lambda_t = v_q$, the permutations w and w' are

$$\begin{aligned} w &= (\hat{1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t + 1}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, \lambda_t, \dots, v_1) \\ w' &= (\hat{1}, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t}, \dots, \overline{\lambda_r + 1}, v_{n+1-k-r}, \dots, \lambda_t + 1, \dots, v_1). \end{aligned}$$

Now, if $\text{type}(w) = 1$ or 2 then $q \leq n-k-r$ and w' still has the same type of w . The permutations are

$$\begin{aligned} w &= (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t + 1}, \dots, \overline{\lambda_r + 1}, \check{1}, v_{n-k-r}, \dots, \lambda_t, \dots, v_1), \\ w' &= (\hat{u}_k, \dots, u_1, \overline{\lambda_1 + 1}, \dots, \overline{\lambda_t}, \dots, \overline{\lambda_r + 1}, \check{1}, v_{n-k-r}, \dots, \lambda_t + 1, \dots, v_1) \end{aligned}$$

where $\check{1}$ changes according to the type of w . Observe that the no matter the type of w , it will only change the values in $(k+t)$ -th and $(n-q+2)$ -th positions. To compute the β -sequences of the Top and Bottom parts, we can use the same ideas as in Case 4 of type C and the results in [19]. Hence,

$$\phi(w) - \phi(w') = (k + \alpha_{n-k-q+2}^T + \lambda_t + 2t - 2)(\varepsilon_{\lambda_t+1} - \varepsilon_{\lambda_t}).$$

The Proposition 6.5 for types B and C can be rephrased for type D and the formula will be $\alpha_{n-k-q+2}^T = -v_q + n - q + 2 - \{j \mid \lambda_j + 1 > v_q\}$, for $1 \leq q \leq n+1-k-r$.

Therefore, $\phi(w) - \phi(w') = (k+n-q+t)(\varepsilon_{\lambda_t+1} - \varepsilon_{\lambda_t})$.

Example 6.9. First of all, consider $\Lambda_1 = [5, 4, 4|7, 6, 1]$ in $\text{OG}(5, 16)$ of type 1 and suppose that we are removing the element $\lambda_3 = 1$. Since the permutation of w_1 can be written, by the row-reading, as $w_1 = \mathbf{s}_1 s_6 s_5 s_4 s_3 s_2 \mathbf{s}_0 \cdots$ then $w'_1 = s_6 s_5 s_4 s_3 s_2 \mathbf{s}_0 \cdots$ and, by Remark 3.3, w'_1 has type 2 and it's partition is $\Lambda' = (5, 4, 4|7, 6]$, where $t = 3$ and $q = 2$. We get: (i) for the top part, $S^T(w_1, w'_1) = 6(\varepsilon_2 - \varepsilon_1)$ and (ii) the sum of the roots is $S^B(w_1, w'_1) = 5(\varepsilon_2 - \varepsilon_1)$. This implies that $\phi(w_1) - \phi(w'_1) = (k + n - q + t)(\varepsilon_2 - \varepsilon_1) = 11(\varepsilon_2 - \varepsilon_1)$. In the other hand, considering $\Lambda_2 = (5, 4, 4|7, 6, 1]$ of type 2, the same argument shows that $\Lambda'_2 = [5, 4, 4|7, 6)$ has type 1 and $\phi(w_2) - \phi(w'_2) = (k + n - q + t)(\varepsilon_2 + \varepsilon_1) = 11(\varepsilon_2 + \varepsilon_1)$.

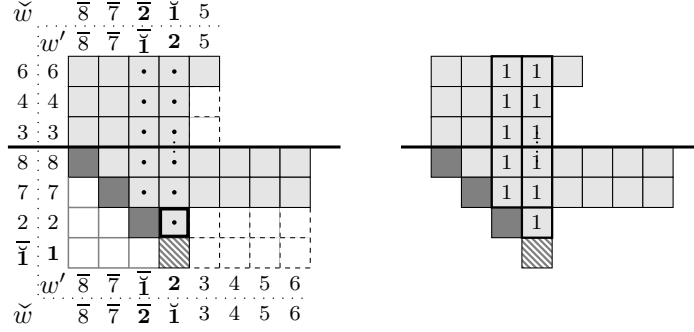


FIGURE 18. $\kappa(w, w')$ for $\Lambda = [5, 4, 4|7, 6, 1]$ and $\Lambda' = [5, 4, 4|7, 6]$.

The above process gives exactly the root $\varepsilon_2 - \varepsilon_1$ as subtraction of β -sequences, the coefficient κ is given in the same way as the λ -removing case of a non-related column for groups of type C and B: fill the diagram in the same way of above example. Adding all this numbers we get exactly the coefficient $\kappa(w, w') = 2k - p + t$.

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